Admissible-Witness Kan Extensions: A Restricted Pointwise Calculus for Controlled Colimits

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Abstract

Kan extensions express extension by universal approximation: a functor $F : A \to C$ is extended along $K : A \to B$ by collecting all witnesses $Ka \to b$ into each object $b \in B$. In many contexts, however, not every witness is intended to participate. One may wish to use only covers, compact approximants, bounded observations, cofibrant replacements, or finite computational probes. This paper introduces admissible-witness Kan extensions, a modest but precise refinement of pointwise left Kan extension in which the full comma functor $(K \downarrow -) : B \to \mathbf{Cat}$ is replaced by a subfunctor of admissible witnesses. The construction is not presented as a new kind of colimit, but as a disciplined way of recording restrictions on pointwise Kan extension.

The main results are threefold. First, admissible-witness Kan extensions are characterized as ordinary colimits weighted by an admissibility subfunctor. Second, they satisfy an adjoint universal property with respect to a category of admissible natural transformations. Third, under a saturation condition on admissibility systems, the construction is closed under composition. A concrete computation in finite sets shows that changing the admissible witnesses changes the resulting extension in a nontrivial and computable way.

The paper situates the construction relative to weighted colimits, sieves, partial colimits, double-categorical Kan extensions, and accessible category theory. The intended contribution is not the invention of a fundamentally new universal construction, but the isolation of a useful restricted pointwise calculus for cases where the admissible indexing data are mathematically meaningful.

1. Introduction

Let

$$K : A \to B, \qquad F : A \to C$$

be functors. When the required colimits exist, the pointwise left Kan extension of F along K is computed by

$$(\mathrm{Lan}_K F)(b) \cong \mathrm{colim}_{(a,\alpha:Ka\to b)\in(K\downarrow b)} F(a).$$

This formula says that the value at b is assembled from all arrows $Ka \to b$. The word “all” is mathematically powerful, but it is also a modeling assumption. In topology one may care only about covering data; in algebra one may care only about finite or finitely presentable approximants; in homotopical algebra one may care only about cofibrant or fibrant witnesses; in computation one may care only about witnesses satisfying resource bounds.

The aim of this paper is to formalize this restriction without pretending that it creates a fundamentally new colimit theory. We replace the full comma functor

$$(K \downarrow -) : B \to \mathbf{Cat}$$

by a subfunctor

$$\mathscr{A} \hookrightarrow (K \downarrow -),$$

where $\mathscr{A}_b \subseteq (K \downarrow b)$ is the category of admissible witnesses into b. We then define

$$(\mathrm{Lan}_K^{\mathscr{A}} F)(b) = \mathrm{colim}_{(a,\alpha)\in\mathscr{A}_b} F(a).$$

The terminology “admissible-witness” is chosen deliberately. We avoid the term “filtered,” since it conflicts with established meanings such as filters in order theory and filtered colimits.

This construction is close to several known ideas. Kan extensions are already understood as partial colimits in the sense explained by Perrone and Tholen, who interpret left Kan extension as replacing parts of a diagram by colimits. Weighted and enriched colimits provide a broad language for changing the indexing mechanism. Double-categorical and virtual-equipment approaches to Kan extensions, including Koudenburg’s work on pointwise and algebraic Kan extensions in double categories, provide still more flexible formal environments. Accessible category theory also makes systematic use of Kan-extension descriptions from small or presentable subcategories; Di Liberti and Loregian’s work studies accessibility and presentability in a 2-categorical setting.

The contribution of this paper is therefore intentionally narrow: It isolates a subfunctorial admissibility condition on comma categories and studies the resulting restricted pointwise Kan-extension calculus, including its adjoint universal property, composition law, and concrete finite computation.

2. The Comma Functor and Admissibility Systems

Let $K : A \to B$ be a functor. Recall that for each $b \in B$, the comma category $(K \downarrow b)$ has objects

$$(a, \alpha : Ka \to b)$$

and morphisms

$$(a, \alpha) \to (a', \alpha')$$

given by morphisms $u : a \to a'$ in A satisfying

$$\alpha' K u = \alpha.$$

The assignment

$$b \mapsto (K \downarrow b)$$

extends to a functor

$$(K \downarrow -) : B \to \mathbf{Cat}$$

by postcomposition: a morphism $f : b \to b'$ sends

$$(a, \alpha : Ka \to b)$$

to

$$(a, f\alpha : Ka \to b').$$

Definition 2.1: Admissibility system

An admissibility system on $K : A \to B$ is a subfunctor

$$\mathscr{A} \hookrightarrow (K \downarrow -)$$

of functors $B \to \mathbf{Cat}$.

Thus, for each $b \in B$, one has a subcategory

$$\mathscr{A}_b \subseteq (K \downarrow b),$$

and for every morphism $f : b \to b'$, postcomposition restricts to a functor

$$f_* : \mathscr{A}_b \to \mathscr{A}_{b'}.$$

Equivalently, if

$$(a, \alpha : Ka \to b) \in \mathscr{A}_b,$$

then

$$(a, f\alpha : Ka \to b') \in \mathscr{A}_{b'}.$$

This closure condition is essential. Without it, the objectwise colimit formula may define objects of C, but it need not define a functor $B \to C$.

Remark 2.2: Why closure under postcomposition matters

Many intuitive restrictions fail to be admissibility systems unless hypotheses are added. For example, if $\mathscr{A}_b$ consists of monomorphisms $Ka \to b$, then closure under postcomposition requires that whenever α is mono, $f\alpha$ is mono. This is not true for arbitrary f. It is true under restrictions such as taking B to be a category whose morphisms preserve the relevant class, or restricting to admissible f’s.

Thus admissibility is not merely a family of subcategories. It is a functorial witness restriction.

3. Admissible-Witness Left Kan Extension

Let C be a category admitting the colimits indexed by the categories $\mathscr{A}_b$.

Definition 3.1

For $F : A \to C$, the admissible-witness left Kan extension of F along K, relative to $\mathscr{A}$, is the functor

$$\mathrm{Lan}_K^{\mathscr{A}} F : B \to C$$

defined by

$(\mathrm{Lan}_K^{\mathscr{A}} F)(b) = \mathrm{colim}_{(a,\alpha)\in\mathscr{A}_b} F(a).$

For $f : b \to b'$, the map

$(\mathrm{Lan}_K^{\mathscr{A}} F)(b) \to (\mathrm{Lan}_K^{\mathscr{A}} F)(b')$

is induced by the functor

$f_* : \mathscr{A}_b \to \mathscr{A}_{b'}.$

Proposition 3.2: Functoriality

If C has the required colimits, then $\mathrm{Lan}_K^{\mathscr{A}} F$ is a well-defined functor $B \to C$. Moreover, the assignment

$F \mapsto \mathrm{Lan}_K^{\mathscr{A}} F$

extends to a functor

$\mathrm{Lan}_K^{\mathscr{A}} : [A, C] \to [B, C].$

Proof

For each b, define

$L(b) = \mathrm{colim}_{\mathscr{A}_b} F.$

For each $f : b \to b'$, admissibility gives

$f_* : \mathscr{A}_b \to \mathscr{A}_{b'}.$

The universal property of the colimit over $\mathscr{A}_b$ induces a unique map

$L(f) : L(b) \to L(b').$

The identity and composition laws follow from the fact that $\mathscr{A}$ is a subfunctor of $(K \downarrow -)$. Hence L is a functor.

If $\theta : F \Rightarrow G$, then for every b, the restriction of θ to $\mathscr{A}_b$ induces

$\mathrm{colim}_{\mathscr{A}_b} F \to \mathrm{colim}_{\mathscr{A}_b} G.$

These maps are natural in b, again by functoriality of $\mathscr{A}$. ∎

4. The Adjoint Universal Property

The preceding construction is most naturally expressed as a left adjoint, but not to the ordinary restriction functor

$K^* : [B, C] \to [A, C]$. Instead, it is left adjoint to a restriction functor after replacing ordinary natural transformations by admissible natural transformations.

Definition 4.1: Admissible natural transformation

Let $F : A \to C$ and $G : B \to C$. An $\mathscr{A}$-admissible natural transformation from F to GK is a family of morphisms

$\theta_{(a,\alpha)} : F(a) \to G(b)$

for every admissible witness

$(a, \alpha : Ka \to b) \in \mathscr{A}_b,$

such that for every morphism

$u : (a, \alpha) \to (a', \alpha')$

in $\mathscr{A}_b$, meaning

$\alpha' K u = \alpha,$

the following equality holds:

$\theta_{(a',\alpha')} \circ F(u) = \theta_{(a,\alpha)}.$

Equivalently, for each b, the family $\theta_{(a,\alpha)}$ is a cocone over the diagram

$\mathscr{A}_b \to A \xrightarrow{F} C$

with vertex $G(b)$.

We write

$\mathrm{Nat}_{\mathscr{A}}(F, GK)$

for the set of all such admissible transformations.

Theorem 4.2: Adjoint form

For every $F : A \to C$ and $G : B \to C$, there is a natural bijection

$[B, C](\mathrm{Lan}_K^{\mathscr{A}} F, G) \cong \mathrm{Nat}_{\mathscr{A}}(F, GK).$

Thus $\mathrm{Lan}_K^{\mathscr{A}}$ is left adjoint to the operation that evaluates a functor $G : B \to C$ on admissible witnesses.

Proof

A natural transformation

$\sigma : \mathrm{Lan}_K^{\mathscr{A}} F \Rightarrow G$

determines, for every admissible witness $(a, \alpha : Ka \to b)$, a composite

$F(a) \to (\mathrm{Lan}_K^{\mathscr{A}} F)(b) \xrightarrow{\sigma_b} G(b).$

These composites form a cocone over $\mathscr{A}_b$, hence define an element of

$\mathrm{Nat}_{\mathscr{A}}(F, GK).$

Conversely, an admissible transformation gives, for each b, a cocone

$F|_{\mathscr{A}_b} \Rightarrow \Delta G(b).$

By the universal property of the colimit, this cocone induces a unique map

$(\mathrm{Lan}_K^{\mathscr{A}} F)(b) \to G(b).$

Naturality in b follows from the postcomposition stability of $\mathscr{A}$. These two constructions are inverse. ∎

5. Relation to Ordinary Kan Extension

Let

$\mathscr{A} = (K \downarrow -).$

Then

$\mathrm{Lan}_K^{\mathscr{A}} F = \mathrm{Lan}_K F$

whenever the ordinary pointwise left Kan extension exists.

This observation is elementary, but important: admissible-witness Kan extension is not a competing construction. It is ordinary pointwise Kan extension with an explicitly restricted indexing functor.

6. A Concrete Computation in Finite Sets

We now give a small example showing that changing admissible witnesses changes the resulting extension.

Let A be the full subcategory of $\mathbf{FinSet}$ on the singleton set 1. Thus A has one object and one morphism. Let

$K : A \hookrightarrow \mathbf{FinSet}$

send the unique object to 1. Let

$F : A \to \mathbf{Set}$

send 1 to a fixed set X.

For a finite set S, the comma category

$(K \downarrow S)$

has as objects functions

$1 \to S.$

Such functions are precisely elements of S. Since A is discrete, $(K \downarrow S)$ is the discrete category on the set S.

Therefore the ordinary left Kan extension is

$$(\mathrm{Lan}_K F)(S) \cong \coprod_{s\in S} X \cong S \times X.$$

Now define an admissibility system $\mathscr{A}$ as follows. For a finite set S, let $\mathscr{A}_S$ be:

$$\mathscr{A}_S = \begin{cases} (K \downarrow S), & |S| \le 2, \\ \varnothing, & |S| > 2. \end{cases}$$

This is not an admissibility system on all of $\mathbf{FinSet}$, because a map from a set of size 2 into a set of size 3 would send admissible witnesses into non-admissible ones. But it becomes one on the wide subcategory $\mathbf{FinSet}_{\le 2}$ consisting of finite sets of cardinality at most 2.

On $\mathbf{FinSet}_{\le 2}$, we have

$$(\mathrm{Lan}_K^{\mathscr{A}} F)(S) \cong S \times X.$$

Now consider a different admissibility system on $\mathbf{FinSet}$. Let $\mathscr{A}_S$ contain only those maps $1 \to S$ whose image is a chosen basepoint of S. This requires working in the category $\mathbf{FinSet}_*$ of finite pointed sets. Let

$$K : A \to \mathbf{FinSet}_*$$

send the unique object to the one-pointed set. For every pointed finite set (S, s_0), there is exactly one admissible witness:

$$1 \to S, \qquad * \mapsto s_0.$$

Thus

$$(\mathrm{Lan}_K^{\mathscr{A}} F)(S, s_0) \cong X.$$

By contrast, the ordinary pointwise left Kan extension gives

$$(\mathrm{Lan}_K F)(S, s_0) \cong S \times X.$$

So the admissible-witness extension forgets all non-basepoint components. It is not merely a rephrasing of the ordinary extension; it is a controlled restriction of the witness category.

7. Sieves, Covers, and Local Witnesses

A particularly important case occurs when each $\mathscr{A}_b$ is a sieve inside $(K \downarrow b)$. In this case admissibility is stable under precomposition inside the comma category. However, an admissibility system requires more: stability under postcomposition in B.

Thus a family of sieves

$$\mathscr{A}_b \subseteq (K \downarrow b)$$

defines an admissibility system only when every morphism $f : b \to b'$ sends admissible arrows into b to admissible arrows into b'.

This condition is not automatic for Grothendieck topologies. Covering families are usually stable under pullback, whereas the present construction requires stability under postcomposition. Therefore the relationship with sheaf theory is real but not identical.

Proposition 7.1

Suppose $K : A \to B$ is fully faithful and $\mathscr{A}_b \subseteq (K \downarrow b)$ is a family of sieves stable under postcomposition in B. Then $\mathrm{Lan}_K^{\mathscr{A}} F$ computes the colimit of F over the admissible local presentations of b.

Proof

Since K is fully faithful, objects of $(K \downarrow b)$ may be viewed as A-objects equipped with maps into b. The subcategory $\mathscr{A}_b$ is exactly the admissible part of this local presentation category. The result follows directly from the defining formula

$$(\mathrm{Lan}_K^{\mathscr{A}} F)(b) = \mathrm{colim}_{\mathscr{A}_b} F.$$

∎

This proposition is deliberately modest. It does not claim that admissible-witness Kan extension is sheafification. Rather, it identifies the precise point at which sheaf-like local data can be used as restricted Kan-extension data.

8. Weighted-Colimit Reformulation

The construction can be expressed in the language of weights. Let

$$P_b : \mathscr{A}_b \to A$$

be the projection

$$(a, \alpha) \mapsto a.$$

Then

$$(\mathrm{Lan}_K^{\mathscr{A}} F)(b) = \mathrm{colim}_{\mathscr{A}_b} F P_b.$$

In enriched language, one would replace $\mathscr{A}_b$ by a suitable weight $W_b : A^{op} \to \mathscr{V}$, obtaining

$$(\mathrm{Lan}_K^{W} F)(b) = W_b \star F.$$

This confirms that admissible-witness Kan extensions are not outside the standard theory of weighted colimits. Their value lies in identifying which weights arise from subfunctorial restrictions of comma categories.

9. Composition of Admissible-Witness Kan Extensions

A weakness of many restricted Kan-extension constructions is that they do not automatically compose. We now give a simple sufficient condition.

Let

$$A \xrightarrow{K} B \xrightarrow{L} D$$

be functors. Let $\mathscr{A} \hookrightarrow (K \downarrow -)$ be an admissibility system on K, and let $\mathscr{B} \hookrightarrow (L \downarrow -)$ be an admissibility system on L.

We want to compare

$$\mathrm{Lan}_L^{\mathscr{B}} \mathrm{Lan}_K^{\mathscr{A}} F$$

with an admissible-witness extension of F along $LK : A \to D$.

For $d \in D$, define a category $(\mathscr{B} \circ \mathscr{A})_d$ whose objects are triples

$$(a, b, \alpha : Ka \to b, \beta : Lb \to d)$$

with

$$(a, \alpha) \in \mathscr{A}_b, \qquad (b, \beta) \in \mathscr{B}_d.$$

A morphism is the evident pair of compatible morphisms in A and B.

There is a comparison functor

$$Q_d : (\mathscr{B} \circ \mathscr{A})_d \to ((LK) \downarrow d)$$

sending

$$(a, b, \alpha, \beta)$$

to

$$(a, \beta L\alpha : LKa \to d).$$

Definition 9.1: Saturated composability

The pair $(\mathscr{A}, \mathscr{B})$ is composition-saturated if the image of each Q_d determines an admissibility system

$$\mathscr{B} \star \mathscr{A} \hookrightarrow ((LK) \downarrow -)$$

and each Q_d is final.

Theorem 9.2: Composition theorem

Assume C has the required colimits and that $(\mathscr{A}, \mathscr{B})$ is composition-saturated. Then there is a natural isomorphism

$$\mathrm{Lan}_L^{\mathscr{B}} \mathrm{Lan}_K^{\mathscr{A}} F \cong \mathrm{Lan}_{LK}^{\mathscr{B}\star\mathscr{A}} F.$$

Proof

For $d \in D$,

$$(\mathrm{Lan}_L^{\mathscr{B}} \mathrm{Lan}_K^{\mathscr{A}} F)(d) = \mathrm{colim}_{(b,\beta)\in\mathscr{B}_d} (\mathrm{Lan}_K^{\mathscr{A}} F)(b).$$

Expanding the inner term gives

$$\operatorname{colim}_{(b,\beta)\in\mathscr{B}_d}\operatorname{colim}_{(a,\alpha)\in\mathscr{A}_b}F(a).$$

By the Fubini theorem for colimits, this is the colimit over the Grothendieck construction of the indexed family

$$(b,\beta)\mapsto\mathscr{A}_b.$$

That Grothendieck construction is precisely $(\mathscr{B}\circ\mathscr{A})_d$. Hence

$$(\mathrm{Lan}_L^{\mathscr{B}}\,\mathrm{Lan}_K^{\mathscr{A}}\,F)(d)\cong\operatorname{colim}_{(\mathscr{B}\circ\mathscr{A})_d}F.$$

Since Q_d is final, this colimit agrees with the colimit over $(\mathscr{B}\star\mathscr{A})_d$. Therefore

$$(\mathrm{Lan}_L^{\mathscr{B}}\,\mathrm{Lan}_K^{\mathscr{A}}\,F)(d)\cong(\mathrm{Lan}_{LK}^{\mathscr{B}\star\mathscr{A}}\,F)(d).$$

Naturality in d follows from the functoriality of the admissibility systems. ∎

This theorem is intentionally stated with a strong hypothesis. The point is not that all admissibility systems compose, but that composition is available when the chosen witness restrictions are saturated under the appropriate two-step presentations.

10. Accessibility Example

Let B be a locally presentable category and let $A=B_\lambda$ be a small full subcategory of λ-presentable objects. Let

$$K:B_\lambda\hookrightarrow B$$

be the inclusion. In standard accessible category theory, accessible functors are often controlled by their behavior on presentable objects; related 2-categorical treatments are studied by Di Liberti and Loregian.

An admissibility system can be defined by taking $\mathscr{A}_b$ to consist of those maps

$$a\to b$$

with $a\in B_\lambda$ satisfying an additional property, such as belonging to a chosen class of embeddings, generators, or presentations.

The admissible-witness extension is then

$$(\mathrm{Lan}_K^{\mathscr{A}}\,F)(b)=\operatorname{colim}_{(a\to b)\in\mathscr{A}_b}F(a).$$

This construction is useful only when $\mathscr{A}$ is stable under postcomposition. For example, if admissible maps are “chosen presentations,” then arbitrary morphisms $b\to b'$ may not preserve admissibility. One must either restrict the morphisms of B, enlarge $\mathscr{A}$, or treat the construction as indexed only over a subcategory of B.

This illustrates a recurring theme: admissibility is mathematically meaningful precisely when it is functorial.

11. Relation to Partial Colimits and Double-Categorical Kan Extensions

Perrone and Tholen describe Kan extensions as partial colimits: one replaces parts of a diagram by colimits while leaving the rest of the structure intact. The present construction can be read as a restricted version of that idea. Instead of taking the partial colimit determined by the entire comma category $(K\downarrow b)$, one takes the partial colimit determined by $\mathscr{A}_b$.

Double categories and virtual double categories offer a more invariant setting for such constructions. Koudenburg’s work develops pointwise and algebraic Kan extensions in double-categorical settings and later extends formal category theory inside augmented virtual double categories. From that viewpoint, an admissibility system should be regarded as a choice of restricted proarrow or restricted module of witnesses. The present paper remains in ordinary category theory, but its definitions are chosen to make this translation plausible.

12. What Is New, and What Is Not

The following are not claimed as new:

taking colimits over subcategories;
weighted colimits;
pointwise Kan-extension formulae;
sieves and covering systems;
double-categorical or equipment-based Kan extensions;
accessible reconstruction from small subcategories.

The contribution is the following package:

define admissible witnesses as a subfunctor of the comma functor;
use that functoriality to obtain a restricted pointwise Kan extension;
formulate the universal property as an adjunction against admissible natural transformations;
identify a sufficient saturation condition for composition;
show by finite-set computation that restricted witnesses can change the extension concretely.

This is a small theory, but it is not empty. It records a precise mathematical situation that occurs whenever one wants Kan-extension-like behavior while refusing to use all comma-category witnesses.

13. Limitations

The construction has several limitations.

First, admissibility systems are restrictive. Many natural-looking classes of witnesses fail to be stable under postcomposition.

Second, the construction is weaker than ordinary Kan extension. Its universal property is relative to admissible witnesses, not absolute among all transformations.

Third, the composition theorem requires a strong saturation condition. Without such a condition, iterated admissible-witness extensions need not agree with a one-step admissible-witness extension.

Fourth, enriched and higher-categorical versions require more care than merely replacing colimits by weighted colimits. One must specify when admissibility data assemble into legitimate enriched weights, modules, or proarrows.

14. Conclusion

Admissible-witness Kan extensions provide a restricted pointwise calculus for Kan-extension-like constructions. Given

$$K:A\to B$$

and an admissibility subfunctor

$$\mathscr{A}\hookrightarrow(K\downarrow -),$$

one defines

$$(\mathrm{Lan}_K^{\mathscr{A}}\,F)(b)=\operatorname{colim}_{(a,\alpha)\in\mathscr{A}_b}F(a).$$

The construction is ordinary in its ingredients but useful in its bookkeeping. It separates two questions that are often conflated:

how should F be extended along K?
which witnesses $Ka\to b$ are allowed to participate?

The answer to the first question is colimit-theoretic. The answer to the second is structural. Admissible-witness Kan extension is the resulting controlled universal construction.

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