A SELF-CONTAINED TÔHOKU THEOREM FOR DIAGRAM CATEGORIES

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Abstract. This note gives a self-contained proof of a classical stability property of Grothendieck abelian categories: if $\mathcal{A}$ is a Grothendieck abelian category and C is a small category, then the functor category $\mathcal{A}^C = \mathrm{Fun}(C, \mathcal{A})$ is again Grothendieck. The result is standard; it appears already in Grothendieck's Tôhoku paper and in modern treatments such as the Stacks Project and Kashiwara–Schapira [3, 7, 4]. The purpose of this exposition is to isolate the elementary mechanism: kernels, cokernels, and filtered colimits are computed objectwise, while a generator is obtained by summing the free diagrams

$$c_! G = \mathrm{Lan}_{i_c}(G),$$

where $i_c : \{c\} \hookrightarrow C$ is the inclusion of an object and G is a generator of $\mathcal{A}$. We also record consequences for injectives, ordinary right derived functors, unbounded derived functors, and the module-category presentation supplied by Gabriel–Popescu.

1. Introduction

Grothendieck's *Sur quelques points d'algèbre homologique* established the categorical setting in which homological algebra can be carried out without reduction to modules: abelian categories with exact filtered colimits and a generator [3]. In modern terminology, these are Grothendieck abelian categories [7, 4].

The elementary but useful observation treated here is that this structure is stable under forming small diagram categories. If C is small and $\mathcal{A}$ is Grothendieck, then

$$\mathcal{A}^C = \mathrm{Fun}(C, \mathcal{A})$$

is again Grothendieck. The same assertion for contravariant diagrams follows immediately by replacing C by C^{op}.

This note is expository. Its contribution is not a new theorem, but a compact proof with all objectwise verifications included, together with the explicit identification of the generator as a sum of left Kan extensions and a concrete computation in the module case.

2. Conventions and Background

A category C is *small* if $\mathrm{Ob}(C)$ is a set and each $\mathrm{Hom}_C(c, d)$ is a set.

An abelian category $\mathcal{A}$ is *AB5* if it has small coproducts and filtered colimits of exact sequences are exact. A *Grothendieck abelian category* is an AB5 abelian category with a generator [3, 7].

A generator of an abelian category $\mathcal{A}$ is an object G such that the functor

$$\mathrm{Hom}_{\mathcal{A}}(G, -) : \mathcal{A} \to \mathbf{Set}$$

is faithful. Equivalently, for every nonzero morphism $f : X \to Y$, there exists $h : G \to X$ such that $fh \neq 0$. In a cocomplete abelian category, this is also equivalent to saying that every object is a quotient of a coproduct of copies of G.

We use covariant functors $C \to \mathcal{A}$. The contravariant version is identical after replacing C by C^{op}.

3. Free Diagrams and Left Kan Extension

Let $c \in C$. Denote by

$$i_c : \{c\} \hookrightarrow C$$

the inclusion of the full subcategory with one object c.

For $A \in \mathcal{A}$, define a functor

$$c_! A : C \to \mathcal{A}$$

by

$$(c_! A)(d) = \bigoplus_{\alpha \in \mathrm{Hom}_C(c,d)} A.$$

This coproduct exists because $\mathrm{Hom}_C(c, d)$ is a set and $\mathcal{A}$ has small coproducts.

For a morphism $\beta : d \to e$, define

$$(c_! A)(\beta) : (c_! A)(d) \to (c_! A)(e)$$

as the morphism induced by the function

$$\mathrm{Hom}_C(c, d) \to \mathrm{Hom}_C(c, e), \qquad \alpha \mapsto \beta\alpha.$$

Thus the summand indexed by $\alpha : c \to d$ is sent identically to the summand indexed by $\beta\alpha : c \to e$.

This is exactly the left Kan extension of A along i_c. In coend notation,

$$(\mathrm{Lan}_{i_c} A)(d) = \int^{x \in \{c\}} \mathrm{Hom}_C(i_c x, d) \cdot A \cong \mathrm{Hom}_C(c, d) \cdot A = \bigoplus_{\alpha : c \to d} A,$$

where $S \cdot A$ denotes the copower $\bigoplus_{s \in S} A$. This is the standard pointwise formula for left Kan extension [5, X.3].

4. The Evaluation Adjunction

For $c \in C$, let

$$\mathrm{ev}_c : \mathcal{A}^C \to \mathcal{A}$$

be evaluation at c.

Lemma 4.1. *For every $c \in C$, the functor*

$$c_! : \mathcal{A} \to \mathcal{A}^C$$

is left adjoint to ev_c. *Explicitly,*

$$\mathrm{Hom}_{\mathcal{A}^C}(c_! A, X) \cong \mathrm{Hom}_{\mathcal{A}}(A, X(c))$$

naturally in $A \in \mathcal{A}$ and $X \in \mathcal{A}^C$.

Proof. Let
$$\eta : A \to X(c)$$
be a morphism in $\mathcal{A}$. We define a natural transformation
$$\tilde{\eta} : c_! A \to X.$$
For each object $d \in C$, the object $(c_! A)(d)$ is
$$\bigoplus_{\alpha : c \to d} A.$$
Define
$$\tilde{\eta}_d : (c_! A)(d) \to X(d)$$
by specifying its restriction to the summand indexed by $\alpha : c \to d$:
$$A \xrightarrow{\eta} X(c) \xrightarrow{X(\alpha)} X(d).$$
Let $\beta : d \to e$. On the summand indexed by $\alpha : c \to d$, the composite
$$(c_! A)(d) \xrightarrow{(c_! A)(\beta)} (c_! A)(e) \xrightarrow{\tilde{\eta}_e} X(e)$$
is
$$A \xrightarrow{\eta} X(c) \xrightarrow{X(\beta\alpha)} X(e).$$
The composite
$$(c_! A)(d) \xrightarrow{\tilde{\eta}_d} X(d) \xrightarrow{X(\beta)} X(e)$$
is
$$A \xrightarrow{\eta} X(c) \xrightarrow{X(\alpha)} X(d) \xrightarrow{X(\beta)} X(e).$$
These are equal because $X(\beta)X(\alpha) = X(\beta\alpha)$. Since the two maps agree on every coproduct summand, $\tilde{\eta}$ is natural.

Conversely, let
$$\theta : c_! A \to X$$
be a natural transformation. At c, the object
$$(c_! A)(c) = \bigoplus_{\gamma \in \mathrm{End}_C(c)} A$$
has a summand indexed by id_c. Define
$$\eta_\theta : A \to X(c)$$
as the composite
$$A \hookrightarrow (c_! A)(c) \xrightarrow{\theta_c} X(c),$$
where the first arrow is the inclusion of the id_c-summand.

Starting with η, forming $\tilde{\eta}$, and then restricting to the id_c-summand gives
$$A \xrightarrow{\eta} X(c) \xrightarrow{X(\mathrm{id}_c)} X(c),$$
which is η.

Starting with θ, form η_θ, and then form $\widetilde{\eta_\theta}$. We show

$$\widetilde{\eta_\theta} = \theta.$$

Let $\alpha : c \to d$. Naturality of θ for α gives

$$X(\alpha)\theta_c = \theta_d(c_! A)(\alpha).$$

The map $(c_! A)(\alpha)$ sends the id_c-summand of $(c_! A)(c)$ identically to the α-summand of $(c_! A)(d)$. Hence the restriction of θ_d to the α-summand is

$$A \xrightarrow{\eta_\theta} X(c) \xrightarrow{X(\alpha)} X(d),$$

which is precisely the restriction of $(\widetilde{\eta_\theta})_d$ to that summand. Therefore $\theta = \widetilde{\eta_\theta}$.

The two assignments are natural in A because precomposition $A' \to A$ affects both sides by precomposition, and they are natural in X because postcomposition $X \to X'$ affects both sides by postcomposition. Hence the bijection is natural in both variables, and $c_! \dashv \mathrm{ev}_c$. □

5. Abelian Structure of Diagram Categories

Proposition 5.1. *If $\mathcal{A}$ is abelian and C is small, then $\mathcal{A}^C$ is abelian. Kernels and cokernels are computed objectwise.*

Proof. Let

$$f : X \to Y$$

be a natural transformation in $\mathcal{A}^C$.

For each $c \in C$, define

$$K(c) = \ker(f_c : X(c) \to Y(c)).$$

Let

$$i_c : K(c) \to X(c)$$

be the kernel inclusion.

For a morphism $\alpha : c \to d$, naturality of f gives

$$Y(\alpha) f_c = f_d X(\alpha).$$

Since $f_c i_c = 0$, we get

$$f_d X(\alpha) i_c = Y(\alpha) f_c i_c = 0.$$

Thus $X(\alpha) i_c$ factors uniquely through $i_d : K(d) \to X(d)$. Define

$$K(\alpha) : K(c) \to K(d)$$

by

$$i_d K(\alpha) = X(\alpha) i_c.$$

The uniqueness in the kernel universal property gives

$$K(\mathrm{id}_c) = \mathrm{id}_{K(c)}$$

and

$$K(\beta\alpha) = K(\beta) K(\alpha)$$

for composable arrows α, β. Therefore $K : C \to \mathcal{A}$ is a functor, and the maps i_c define a natural transformation

$$i : K \to X.$$

We show that i is the kernel of f in $\mathcal{A}^C$. Let

$$h : Z \to X$$

be a natural transformation with $fh = 0$. For each c, since

$$f_c h_c = 0,$$

there is a unique morphism

$$\overline{h}_c : Z(c) \to K(c)$$

with

$$i_c \overline{h}_c = h_c.$$

For $\alpha : c \to d$, compare

$$K(\alpha)\overline{h}_c \quad \text{and} \quad \overline{h}_d Z(\alpha).$$

After composing with i_d, the first becomes

$$i_d K(\alpha)\overline{h}_c = X(\alpha) i_c \overline{h}_c = X(\alpha) h_c.$$

The second becomes

$$i_d \overline{h}_d Z(\alpha) = h_d Z(\alpha).$$

These are equal by naturality of h. Since i_d is monic,

$$K(\alpha)\overline{h}_c = \overline{h}_d Z(\alpha).$$

Thus the $\overline{h}_c$ define a natural transformation

$$\overline{h} : Z \to K.$$

It is unique because it is unique at every object. Hence $K \to X$ is the kernel of f.

The cokernel is dual. Define

$$Q(c) = \text{coker}(f_c : X(c) \to Y(c))$$

with quotient map

$$q_c : Y(c) \to Q(c).$$

For $\alpha : c \to d$, the equality

$$q_d Y(\alpha) f_c = q_d f_d X(\alpha) = 0$$

implies that $q_d Y(\alpha)$ factors uniquely through q_c, giving a morphism

$$Q(\alpha) : Q(c) \to Q(d)$$

with

$$Q(\alpha) q_c = q_d Y(\alpha).$$

Again uniqueness gives functoriality, and the objectwise universal property gives that Q is the cokernel of f in $\mathcal{A}^C$.

Finally, the canonical morphism

$$\text{coim}(f) \to \text{im}(f)$$

is computed objectwise. Since $\mathcal{A}$ is abelian, it is an isomorphism at every object of C, and hence is an isomorphism in $\mathcal{A}^C$. Therefore $\mathcal{A}^C$ is abelian. □

6. Colimits and AB5

Proposition 6.1. *If $\mathcal{A}$ is cocomplete and C is small, then $\mathcal{A}^C$ is cocomplete and colimits are computed objectwise.*

Proof. Let

$$D : I \to \mathcal{A}^C$$

be a small diagram.

For each $c \in C$, define

$$L(c) = \mathrm{colim}_{i\in I} D(i)(c).$$

For a morphism $\alpha : c \to d$, the maps

$$D(i)(\alpha) : D(i)(c) \to D(i)(d)$$

form a natural transformation of I-diagrams in $\mathcal{A}$:

$$D(-)(c) \to D(-)(d).$$

Taking colimits gives

$$L(\alpha) : L(c) \to L(d).$$

Functoriality follows from the uniqueness part of the universal property of colimits. Hence $L : C \to \mathcal{A}$ is a functor.

For each $i \in I$, the colimit maps

$$D(i)(c) \to L(c)$$

are natural in c, so they define a natural transformation

$$D(i) \to L.$$

Thus L is a cocone over D.

Let $M \in \mathcal{A}^C$ be another cocone target. For each c, the cocone maps

$$D(i)(c) \to M(c)$$

induce a unique morphism

$$u_c : L(c) \to M(c).$$

For $\alpha : c \to d$, the equality

$$M(\alpha)u_c = u_d L(\alpha)$$

holds because, after precomposing with every colimit map $D(i)(c) \to L(c)$, both sides equal

$$D(i)(c) \xrightarrow{D(i)(\alpha)} D(i)(d) \to M(d).$$

Therefore the u_c assemble into a natural transformation

$$u : L \to M.$$

The uniqueness of u follows objectwise. Hence L is the colimit of D in $\mathcal{A}^C$. □

Proposition 6.2. *If $\mathcal{A}$ is AB5 and C is small, then $\mathcal{A}^C$ is AB5.*

Proof. By Proposition 6.1, $\mathcal{A}^C$ has small coproducts and filtered colimits.

Let I be filtered, and let

$$0 \to X_i \to Y_i \to Z_i \to 0$$

be a filtered diagram of short exact sequences in $\mathcal{A}^C$.

By Proposition 5.1, exactness in $\mathcal{A}^C$ is objectwise exactness. Therefore, for every $c \in C$,

$$0 \to X_i(c) \to Y_i(c) \to Z_i(c) \to 0$$

is a filtered diagram of short exact sequences in $\mathcal{A}$.

Since $\mathcal{A}$ is AB5, the filtered colimit sequence

$$0 \to \operatorname{colim}_i X_i(c) \to \operatorname{colim}_i Y_i(c) \to \operatorname{colim}_i Z_i(c) \to 0$$

is exact in $\mathcal{A}$.

By Proposition 6.1, this is the value at c of the colimit sequence in $\mathcal{A}^C$. Hence the colimit sequence is exact at every c. Using Proposition 5.1 again, it is exact in $\mathcal{A}^C$. Therefore $\mathcal{A}^C$ satisfies AB5. □

7. The Generator

Theorem 7.1. *Let $\mathcal{A}$ be a cocomplete abelian category with generator G, and let C be small. Then*

$$G_C = \bigoplus_{c \in \mathrm{Ob}(C)} c_! G$$

is a generator of $\mathcal{A}^C$.

Proof. The coproduct exists because $\mathrm{Ob}(C)$ is a set and $\mathcal{A}^C$ is cocomplete.

We prove that

$$\mathrm{Hom}_{\mathcal{A}^C}(G_C, -)$$

is faithful.

Let

$$f : X \to Y$$

be a nonzero morphism in $\mathcal{A}^C$. Then there exists $d \in C$ such that

$$f_d : X(d) \to Y(d)$$

is nonzero in $\mathcal{A}$.

Since G is a generator of $\mathcal{A}$, there exists

$$g : G \to X(d)$$

such that

$$f_d g \neq 0.$$

By Lemma 4.1, g corresponds to a natural transformation

$$\tilde{g} : d_! G \to X.$$

Let

$$\iota_d : d_! G \to G_C$$

be the coproduct inclusion. By the universal property of the coproduct, there is a morphism

$$h : G_C \to X$$

whose restriction to the summand $d_!G$ is $\tilde{g}$ and whose restriction to every other summand is 0.

Then $fh \neq 0$. Indeed, the restriction of fh to the summand $d_!G$ is

$$f\tilde{g} : d_!G \to Y.$$

Under the adjunction $d_! \dashv \mathrm{ev}_d$, this corresponds to

$$G \xrightarrow{g} X(d) \xrightarrow{f_d} Y(d),$$

which is nonzero. Thus $f\tilde{g} \neq 0$, and therefore $fh \neq 0$.

Hence every nonzero morphism in $\mathcal{A}^C$ is detected by a morphism from G_C. Therefore $\mathrm{Hom}_{\mathcal{A}^C}(G_C, -)$ is faithful, and G_C is a generator. □

Corollary 7.2. *Every object $X \in \mathcal{A}^C$ is a quotient of a coproduct of copies of G_C.*

Proof. Since G_C is a generator, for each object X consider the canonical morphism

$$\Phi_X : \bigoplus_{\varphi \in \mathrm{Hom}_{\mathcal{A}^C}(G_C, X)} G_C \longrightarrow X$$

whose restriction to the summand indexed by φ is φ.

We show that Φ_X is an epimorphism. If it were not, its cokernel

$$q : X \to Q$$

would be nonzero. Since G_C is a generator, there would exist

$$\psi : G_C \to X$$

such that

$$q\psi \neq 0.$$

But ψ is one of the summands in the definition of Φ_X, so $q\psi = 0$, a contradiction. Hence Φ_X is epi. □

8. The Tôhoku Diagram Theorem

Theorem 8.1 (Tôhoku diagram theorem)**.** *Let $\mathcal{A}$ be a Grothendieck abelian category and let C be a small category. Then*

$$\mathcal{A}^C$$

is a Grothendieck abelian category.

If G is a generator of $\mathcal{A}$, then

$$G_C = \bigoplus_{c \in \mathrm{Ob}(C)} c_!G$$

is a generator of $\mathcal{A}^C$.

Proof. Since $\mathcal{A}$ is Grothendieck, it is abelian, cocomplete, AB5, and has a generator G.

By Proposition 5.1, $\mathcal{A}^C$ is abelian.
By Proposition 6.1, $\mathcal{A}^C$ is cocomplete.
By Proposition 6.2, $\mathcal{A}^C$ is AB5.
By Theorem 7.1,

$$G_C = \bigoplus_{c \in \mathrm{Ob}(C)} c_! G$$

is a generator of $\mathcal{A}^C$.

Therefore $\mathcal{A}^C$ is an AB5 abelian category with a generator. Hence $\mathcal{A}^C$ is Grothendieck. □

9. Injectives and Derived Functors

Corollary 9.1. *The category $\mathcal{A}^C$ has enough injectives.*

Proof. By Theorem 8.1, $\mathcal{A}^C$ is Grothendieck. Grothendieck categories have enough injectives [3, 7]. Therefore $\mathcal{A}^C$ has enough injectives. □

Corollary 9.2. *Let*

$$F : \mathcal{A}^C \to \mathcal{B}$$

be a left exact functor into an abelian category $\mathcal{B}$. Then the ordinary right derived functors

$$R^n F : \mathcal{A}^C \to \mathcal{B}$$

exist for all $n \geq 0$.

Proof. By Corollary 9.1, every object $X \in \mathcal{A}^C$ admits an injective resolution

$$0 \to X \to I^0 \to I^1 \to I^2 \to \cdots.$$

Apply F to the deleted resolution:

$$0 \to F(I^0) \to F(I^1) \to F(I^2) \to \cdots.$$

Define

$$R^n F(X) = H^n(F(I^\bullet)).$$

The comparison theorem for injective resolutions gives canonical isomorphisms between the cohomology objects obtained from any two injective resolutions. Functoriality follows by lifting morphisms to chain maps between injective resolutions, unique up to homotopy. Thus $R^n F$ is a well-defined functor.

The usual horseshoe and connecting-homomorphism argument gives long exact sequences associated to short exact sequences in $\mathcal{A}^C$. Hence the $R^n F$ are the ordinary right derived functors of F. □

Corollary 9.3. *If $\mathcal{B}$ is also a Grothendieck abelian category and*

$$F : \mathcal{A}^C \to \mathcal{B}$$

is left exact, then F admits an unbounded total right derived functor

$$RF : D(\mathcal{A}^C) \to D(\mathcal{B}).$$

Proof. By Theorem 8.1, $\mathcal{A}^C$ is Grothendieck. In a Grothendieck abelian category, every complex admits a quasi-isomorphism into a K-injective complex; this is Spaltenstein's theorem in the Grothendieck setting, in the form used in modern derived-category treatments [6, 4].

For a complex $X^\bullet \in D(\mathcal{A}^C)$, choose a quasi-isomorphism

$$X^\bullet \to I^\bullet$$

with $I^\bullet$ K-injective. Define

$$RF(X^\bullet) = F(I^\bullet).$$

If $X^\bullet \to J^\bullet$ is another K-injective replacement, the comparison theorem for K-injective resolutions gives an isomorphism

$$F(I^\bullet) \cong F(J^\bullet)$$

in $D(\mathcal{B})$. Hence RF is well-defined on the derived category and is functorial. □

10. The Module Case and the Category Algebra

The abstract generator G_C becomes especially concrete when $\mathcal{A} = \mathrm{Mod}(k)$, where k is a ring and $G = k$.

Assume C is small. Let kC denote the k-linear category generated by C: its objects are those of C, and

$$kC(c, d) = k[\mathrm{Hom}_C(c, d)]$$

is the free k-module on $\mathrm{Hom}_C(c, d)$, with composition extended k-bilinearly.

A covariant functor

$$C \to \mathrm{Mod}(k)$$

is equivalently a k-linear functor

$$kC \to \mathrm{Mod}(k).$$

Thus

$$\mathrm{Mod}(k)^C \simeq \mathrm{Mod}(kC),$$

where the right-hand side means the category of left modules over the small k-linear category kC.

Proposition 10.1. *Let*

$$P_c = c_! k.$$

Then P_c is the representable left kC-module

$$P_c(d) = kC(c, d).$$

Consequently,

$$G_C = \bigoplus_{c \in \mathrm{Ob}(C)} P_c$$

is the usual sum of representable projective generators of $\mathrm{Mod}(kC)$.

Proof. By definition,

$$(c_!k)(d) = \bigoplus_{\alpha \in \mathrm{Hom}_C(c,d)} k.$$

This is exactly the free k-module on $\mathrm{Hom}_C(c, d)$, namely

$$k[\mathrm{Hom}_C(c, d)] = kC(c, d).$$

For a morphism $\beta : d \to e$, the map

$$(c_!k)(d) \to (c_!k)(e)$$

sends the basis vector $[\alpha]$ to $[\beta\alpha]$. This is exactly the action of the representable $kC(c, -)$ on morphisms. Hence

$$c_!k \cong kC(c, -).$$

The direct sum over all c is therefore the direct sum of all representable left kC-modules, which is the standard projective generator of the module category over the small k-linear category kC. □

Proposition 10.2. *There is a natural identification*

$$\mathrm{End}_{\mathrm{Mod}(k)^C}(G_C) \cong \bigoplus_{c,d \in \mathrm{Ob}(C)} kC(c, d)$$

with multiplication given by matrix multiplication using composition in kC, whenever $\mathrm{Ob}(C)$ *is finite.*

For infinite C, the same formula gives the ring of column-finite matrices with entries in $kC(c, d)$, equivalently the endomorphism ring of the direct sum of representables.

Proof. Write

$$G_C = \bigoplus_c P_c, \qquad P_c = kC(c, -).$$

For representables, the enriched Yoneda lemma gives

$$\mathrm{Hom}(P_c, P_d) \cong P_d(c) = kC(d, c).$$

An endomorphism of the finite direct sum $\bigoplus_c P_c$ is a matrix whose (d, c)-entry is a morphism

$$P_c \to P_d,$$

hence an element of

$$kC(d, c).$$

Equivalently, after relabelling indices, the endomorphism ring is the matrix ring built from the morphism modules of kC.

Composition of endomorphisms is matrix multiplication, and the product of entries is composition in kC. For infinite C, morphisms out of a direct sum are determined by compatible families on summands, giving the usual column-finiteness condition for endomorphisms of a direct sum. □

Remark 10.3. When C has one object and endomorphism monoid M, the category algebra kC is the monoid algebra $k[M]$, and $\mathrm{Mod}(k)^C$ is the category of k-modules equipped with an M-action.

When C is a finite category, the ring in Proposition 10.2 is the ordinary category algebra of C, up to the conventional choice of left-versus-right module orientation.

11. Gabriel–Popescu Presentation

The Gabriel–Popescu theorem says that every Grothendieck abelian category is an exact localization of a module category [2, 7].

Corollary 11.1. *Let $\mathcal{A}$ be Grothendieck and C small. Then $\mathcal{A}^C$ is an exact localization of a module category. More precisely, if G_C is the generator of Theorem 7.1, then there is a ring with several objects, or equivalently a small preadditive category $\mathcal{R}_C$, and an exact functor*

$$Q : \mathrm{Mod}(\mathcal{R}_C) \to \mathcal{A}^C$$

which admits a fully faithful right adjoint.

Proof. By Theorem 8.1, $\mathcal{A}^C$ is Grothendieck. By Gabriel–Popescu, every Grothendieck abelian category is an exact localization of a module category over a small preadditive category determined by a set of generators [2]. Applying this theorem to $\mathcal{A}^C$, with the generator G_C from Theorem 7.1, gives the asserted exact localization. □

12. Literature Positioning

The theorem proved here is classical. Grothendieck's Tôhoku paper already treats categories of diagrams as preserving the relevant hypotheses [3]. The Stacks Project records modern statements on Grothendieck categories, injectives, and derived categories [7]. Kashiwara–Schapira give a systematic treatment of abelian categories, Grothendieck categories, and unbounded derived functors [4]. The Kan-extension formula used for $c_!$ is the standard pointwise left Kan extension formula [5]. Borceux gives a broad categorical treatment of locally presentable and Grothendieck-style categories [1].

The value of the present note is pedagogical: it separates the proof into the three elementary ingredients–objectwise abelian structure, objectwise AB5, and the free-diagram generator–and records the module-category interpretation of the generator.

References

[1] F. Borceux, *Handbook of Categorical Algebra 2: Categories and Structures*, Encyclopedia of Mathematics and its Applications, vol. 51, Cambridge University Press, Cambridge, 1994.

[2] P. Gabriel and N. Popescu, Caractérisation des catégories abéliennes avec générateurs et limites inductives exactes, *C. R. Acad. Sci. Paris* **258** (1964), 4188–4190.

[3] A. Grothendieck, Sur quelques points d'algèbre homologique, *Tohoku Math. J. (2)* **9** (1957), 119–221.

[4] M. Kashiwara and P. Schapira, *Categories and Sheaves*, Grundlehren der mathematischen Wissenschaften, vol. 332, Springer, Berlin, 2006.

[5] S. Mac Lane, *Categories for the Working Mathematician*, 2nd ed., Graduate Texts in Mathematics, vol. 5, Springer, New York, 1998.

[6] N. Spaltenstein, Resolutions of unbounded complexes, *Compositio Math.* **65** (1988), no. 2, 121–154.

[7] The Stacks Project Authors, *The Stacks Project*, available at `https://stacks.math.columbia.edu`; see especially Tags 07D7 and 0D5V.

[8] C. A. Weibel, *An Introduction to Homological Algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994.