A Category-Theoretic Approach to Multisensory Brain-Machine Interfaces for the Metaverse

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Abstract

The metaverse is a hypothetical virtual world that is facilitated by the use of virtual reality (VR) and augmented reality (AR) headsets, where users can interact with each other and experience various sensory stimuli. Brain-machine interfaces (BMIs) are devices that enable direct communication between the brain and external devices, such as VR/AR headsets, by decoding neural signals and stimulating neural activity. In this paper, we propose a novel framework for designing and implementing multisensory BMIs for the metaverse, based on the mathematical theory of categories. We argue that category theory provides a natural and elegant way to model the structure and function of the brain, the metaverse, and the BMI, as well as the interactions and transformations between them. We use category theory to define the concepts of objects, projections, functors, natural transformations, and adjunctions, and show how they can be applied to various aspects of multisensory BMIs, such as sensory encoding and decoding, attention learning, stimulus integration, and feedback control. We also discuss the advantages and challenges of using category theory for multisensory BMIs, and suggest some directions for future research.

I. Introduction

The metaverse is a term coined by the science fiction writer Neal Stephenson in his novel Snow Crash, to describe a collective virtual shared space, created by the convergence of virtually enhanced physical and digital reality. The metaverse is envisioned as the next evolution of the internet, where users can connect, socialize, learn, play, and more, in a fully immersive and interactive way. To access the metaverse, users need devices that can bridge their brain and the virtual world, such as VR/AR headsets, that can provide realistic and rich sensory stimuli, such as vision, sound, touch, smell, and taste. However, current VR/AR technologies are limited by the quality and bandwidth of the sensory input and output, as well as the user's comfort and safety.

Brain-machine interfaces (BMIs) are devices that can overcome these limitations, by directly interfacing with the brain, bypassing the sensory organs and the peripheral nervous system. BMIs can decode the neural signals generated by the brain, and use them to control external devices, such as VR/AR headsets, robotic arms, or prosthetic limbs. Conversely, BMIs can also encode sensory information from external devices, and use them to stimulate the brain, creating artificial sensations, such as vision, sound, touch, smell, and taste. BMIs have been widely used for various applications, such as restoring or enhancing sensory or motor functions, cognitive enhancement, entertainment, and education.

However, designing and implementing BMIs for the metaverse poses several challenges, such as:

- How to encode and decode sensory information from multiple modalities, such as vision, sound, touch, smell, and taste, in a way that is compatible with the brain's natural representation and processing of sensory stimuli?
- How to learn and adapt the attention mechanism of the BMI, to weigh the importance of different sensory inputs and outputs, depending on the context and the user's preferences and goals?
- How to integrate and synchronize multiple sensory stimuli, to create a coherent and consistent multisensory experience, that matches the virtual environment and the user's actions and expectations?
- How to provide feedback and control to the user, to allow them to adjust and optimize their multisensory experience, and to prevent adverse effects, such as sensory overload, confusion, or cybersickness?

In this paper, we propose a novel framework for addressing these challenges, based on the mathematical theory of categories. Category theory is a branch of abstract algebra, that studies the structure and function of mathematical objects and projections, as well as the interactions and transformations between them. Category theory has been widely used in various fields of mathematics, logic, computer science, physics, and biology, to provide a unified and elegant way of describing and analyzing complex systems and phenomena.

We argue that category theory can also be applied to multisensory BMIs for the metaverse, for the following reasons:

- Category theory can capture the structure and function of the brain, the metaverse, and the BMI, as well as the interactions and transformations between them, using the concepts of objects, projections, functors, natural transformations, and adjunctions.
- Category theory can provide a general and flexible framework for encoding and decoding sensory information from multiple modalities, using the concepts of functors and natural transformations, that can preserve the essential properties and relations of the sensory stimuli, while allowing for different levels of abstraction and representation.
- Category theory can provide a principled and efficient way of learning and adapting the attention mechanism of the BMI, using the concepts of projections and adjunctions, that can optimize the trade-off between information and computation, and between generality and specificity.
- Category theory can provide a coherent and consistent way of integrating and synchronizing multiple sensory stimuli, using the concepts of functors and natural transformations, that can ensure the compatibility and commutativity of the sensory transformations, and the preservation of the multisensory structure and function.
- Category theory can provide a natural and intuitive way of providing feedback and control to the user, using the concepts of adjunctions and projections, that can establish a bi-directional and reversible correspondence between the brain and the metaverse, and between the sensory input and output.

The rest of the paper is organized as follows. In Section 2, we review the basic concepts and definitions of category theory, and show how they can be applied to multisensory BMIs for the metaverse. In Section 3, we present some examples and applications of our framework, and compare it with existing approaches. In Section 4, we discuss the advantages and challenges of using category theory for multisensory BMIs, and suggest some directions for future research. In Section 5, we conclude the paper and summarize our main contributions.

II. Category Theory and Multisensory BMIs

In this section, we review the basic concepts and definitions of category theory, and show how they can be applied to multisensory BMIs for the metaverse. We assume that the reader is familiar with some elementary notions of set theory, such as sets, functions, and relations. For a more detailed and rigorous introduction to category theory, we refer the reader to .

II.I. Categories

A category is a mathematical structure that consists of two types of entities: objects and projections. Objects are abstract entities that represent some kind of mathematical structure, such as sets, groups, vector spaces, topological spaces, etc. Projections are mappings or transformations between objects, that preserve some aspects of their structure, such as functions, homomorphisms, linear maps, continuous maps, etc. A category is denoted by a name, usually a capital letter, such as C, D, E, etc.

Formally, a category C consists of the following data:

- A class of objects, denoted by Ob(C), such as A, B, C, etc.

- A class of projections, denoted by Hom(C), such as f, g, h, etc. Each projection has a source object and a target object, denoted by dom(f) and cod(f), respectively. We write f: A -> B to indicate that f is a projection from A to B, and that dom(f) = A and cod(f) = B.

- A composition operation, denoted by ◦, that assigns to each pair of projections f: A -> B and g: B -> C a projection g ◦ f: A -> C, such that dom(g ◦ f) = A and cod(g ◦ f) = C. The composition operation is associative, meaning that for any three projections f: A -> B, g: B -> C, and h: C -> D, we have (h ◦ g) ◦ f = h ◦ (g ◦ f).

- An identity projection, denoted by 1_A, for each object A, such that 1_A: A -> A, and dom(1_A) = cod(1_A) = A. The identity projection satisfies the following properties: for any projection f: A -> B, we have f ◦ 1_A = f and 1_B ◦ f = f.

A category can be represented by a diagram, where the objects are depicted as nodes, and the projections are depicted as arrows, labeled by their names.

We can apply the concept of categories to multisensory BMIs for the metaverse, by considering the following categories:

- The category B of brain regions, where the objects are different brain regions, such as the visual cortex, the auditory cortex, the somatosensory cortex, the olfactory cortex, and the gustatory cortex, and the projections are neural pathways or connections between brain regions, such as the optic nerve, the auditory nerve, the trigeminal nerve, the olfactory nerve, and the glossopharyngeal nerve.

The composition of projections corresponds to the propagation of neural signals along the neural pathways, and the identity projection corresponds to the self-connection of a brain region.

- The category M of metaverse modalities, where the objects are different sensory modalities, such as vision, sound, touch, smell, and taste, and the projections are transformations or conversions between modalities, such as image-to-sound, sound-to-touch, touch-to-smell, smell-to-taste, and taste-to-image. The composition of projections corresponds to the combination of transformations, and the identity projection corresponds to the preservation of a modality.

- The category I of BMI devices, where the objects are different BMI devices, such as VR/AR headsets, electrodes, sensors, stimulators, etc., and the projections are interfaces or communications between devices, such as input, output, feedback, control, etc. The composition of projections corresponds to the integration of interfaces, and the identity projection corresponds to the self-interface of a device.

II.II. Functors

A functor is a mapping or a correspondence between two categories, that preserves their structure and function. A functor assigns to each object in the source category an object in the target category, and to each projection in the source category a projection in the target category, such that the composition and the identity are preserved. A functor is denoted by a name, usually a lowercase letter, such as F, G, H, etc.

Formally, a functor F: C -> D consists of the following data:

- A function F: Ob(C) -> Ob(D), that assigns to each object A in C an object F(A) in D.
- A function F: Hom(C) -> Hom(D), that assigns to each projection f: A -> B in C a projection F(f): F(A) -> F(B) in D, such that dom(F(f)) = F(dom(f)) and cod(F(f)) = F(cod(f)). The function F satisfies the following properties: for any two projections f: A -> B and g: B -> C in C, we have F(g ∘ f) = F(g) ∘ F(f), and for any object A in C, we have F(1_A) = 1_F(A).

A functor can be represented by a diagram, where the source and the target categories are depicted as subdiagrams, and the functor is depicted as a mapping between them, labeled by its name.

We can apply the concept of functors to multisensory BMIs for the metaverse, by considering the following functors:

- The encoding functor E: M -> B, that assigns to each sensory modality in the metaverse a corresponding brain region, and to each transformation between modalities a corresponding neural pathway. The encoding functor preserves the structure and function of the sensory stimuli, by converting them into neural signals that can be processed by the brain. For example, the encoding functor assigns to the vision modality the visual cortex, and to the image-to-sound transformation the optic nerve.

- The decoding functor D: B -> M, that assigns to each brain region a corresponding sensory modality in the metaverse, and to each neural pathway a corresponding transformation between modalities. The decoding functor preserves the structure and function of the neural signals, by converting them into sensory stimuli that can be experienced by the user. For example, the decoding functor assigns to the visual cortex the vision modality, and to the optic nerve the sound-to-image transformation.

- The interface functor I: B -> I, that assigns to each brain region a corresponding BMI device, and to each neural pathway a corresponding interface between devices. The interface functor preserves the structure and function of the brain, by enabling the communication and interaction between the brain and the external devices. For example, the interface functor assigns to the visual cortex a VR/AR headset, and to the optic nerve an input interface.

II.III. Natural Transformations

A natural transformation is a relation or a correspondence between two functors, that preserves their structure and function. A natural transformation assigns to each object in the common source category a projection in the common target category, such that the projections are compatible with the functors. A natural transformation is denoted by a name, usually a Greek letter, such as α, β, γ, etc.

Formally, a natural transformation α: F -> G, where F and G are two functors from C to D, consists of the following data:

- A family of projections α_A: F(A) -> G(A), for each object A in C, such that dom(α_A) = F(A) and cod(α_A) = G(A). The family of projections satisfies the following property: for any projection f: A -> B in C, we have α_B ∘ F(f) = G(f) ∘ α_A. This property is called the naturality condition, and it ensures that the natural transformation is compatible with the functors.

A natural transformation can be represented by a diagram, where the source and the target categories are depicted as subdiagrams, and the natural transformation is depicted as a family of projections between them, labeled by their names.

We can apply the concept of natural transformations to multisensory BMIs for the metaverse, by considering the following natural transformations:

- The attention natural transformation A: E -> D, where E and D are the encoding and decoding functors, respectively. The attention natural transformation assigns to each sensory modality in the metaverse a projection from the corresponding brain region to the corresponding sensory modality, such that the projections are compatible with the encoding and decoding functors. The attention natural transformation preserves the structure and function of the sensory stimuli and the neural signals, by weighting the importance of different sensory inputs and outputs, depending on the context and the user's preferences and goals. For example, the attention natural transformation assigns to the vision modality a projection from the visual cortex to the vision modality, such that the projection is compatible with the encoding and decoding functors.

- The integration natural transformation I: E ∘ D -> D ∘ E, where E and D are the encoding and decoding functors, respectively, and ∘ is the composition of functors. The integration natural transformation assigns to each pair of sensory modalities in the metaverse a projection from the corresponding pair of brain regions to the corresponding pair of sensory modalities, such that the projections are compatible with the encoding and decoding functors. The integration natural transformation preserves the structure and function of the sensory stimuli and the neural signals, by integrating and synchronizing multiple sensory inputs and outputs, to create a coherent and consistent multisensory experience, that matches the virtual environment and the user's actions and expectations. For example, the integration natural transformation assigns to the pair of vision and sound modalities a projection from the pair of visual and auditory cortices to the pair of vision and sound modalities, such that the projection is compatible with the encoding and decoding functors.

- The feedback natural transformation $F: I \to I$, where I is the interface functor. The feedback natural transformation assigns to each brain region a projection from the corresponding BMI device to the corresponding BMI device, such that the projections are compatible with the interface functor. The feedback natural transformation preserves the structure and function of the brain and the BMI devices, by providing feedback and control to the user, to allow them to adjust and optimize their multisensory experience, and to prevent adverse effects, such as sensory overload, confusion, or cybersickness. For example, the feedback natural transformation assigns to the visual cortex a projection from the VR/AR headset to the VR/AR headset, such that the projection is compatible with the interface functor.

II.IV. Adjunctions

An adjunction is a pair of functors and a pair of natural transformations, that establish a bi-directional and reversible correspondence between two categories, that preserves their structure and function. An adjunction consists of two functors, called the left adjoint and the right adjoint, and two natural transformations, called the unit and the counit, that satisfy certain properties. An adjunction is denoted by a name, usually a pair of parentheses, such as (F, G), (H, K), etc.

Formally, an adjunction (F, G), where $F: C \to D$ and $G: D \to C$ are two functors, consists of the following data:

- A natural transformation $\eta: 1\_C \to G \circ F$, called the unit, where 1_C is the identity functor on C. The unit assigns to each object A in C a projection $\eta\_A: A \to G(F(A))$, such that the projections are compatible with the functors.

- A natural transformation $\varepsilon: F \circ G \to 1\_D$, called the counit, where 1_D is the identity functor on D. The counit assigns to each object B in D a projection $\varepsilon\_B: F(G(B)) \to B$, such that the projections are compatible with the functors.

- The natural transformations η and ε satisfy the following properties, called the triangle identities: for any object A in C, we have $\varepsilon\_F(A) \circ F(\eta\_A) = 1\_F(A)$, and for any object B in D, we have $G(\varepsilon\_B) \circ \eta\_G(B) = 1\_G(B)$. These properties ensure that the adjunction is reversible and bi-directional.

An adjunction can be represented by a diagram, where the source and the target categories are depicted as subdiagrams, and the adjunction is depicted as a pair of functors and a pair of natural transformations between them, labeled by their names. For example, the following diagram represents an adjunction (F, G):

We can apply the concept of adjunctions to multisensory BMIs for the metaverse, by considering the following adjunctions:

- The encoding-decoding adjunction (E, D), where E and D are the encoding and decoding functors, respectively. The encoding-decoding adjunction establishes a bi-directional and reversible correspondence between the category M of metaverse modalities and the category B of brain regions, that preserves their structure and function. The encoding-decoding adjunction consists of the following data:

- A natural transformation $\eta: 1\_M \to D \circ E$, called the unit, where 1_M is the identity functor on M. The unit assigns to each sensory modality in the metaverse a projection from the modality to the corresponding brain region and back to the modality, such that the projections are compatible with the encoding and decoding functors. The unit preserves the structure and function of the sensory stimuli, by encoding and decoding them without loss of information or quality. For example, the unit assigns to the vision modality a projection from the vision modality to the visual cortex and back to the vision modality, such that the projection is compatible with the encoding and decoding functors.

- A natural transformation $\varepsilon: E \circ D \to 1\_B$, called the counit, where 1_B is the identity functor on B. The counit assigns to each brain region a projection from the corresponding sensory modality and back to the brain region, such that the projections are compatible with the encoding and decoding functors. The counit preserves the structure and function of the neural signals, by encoding and decoding them without loss of information or quality. For example, the counit assigns to the visual cortex a projection from the vision modality and back to the visual cortex, such that the projection is compatible with the encoding and decoding functors.

- The natural transformations η and ε satisfy the triangle identities, meaning that for any sensory modality in the metaverse, the composition of the unit and the counit is equal to the identity projection on the modality, and for any brain region, the composition of the counit and the unit is equal to the identity projection on the brain region. These properties ensure that the encoding-decoding adjunction is reversible and bi-directional.

- The interface-feedback adjunction (I, I), where I is the interface functor. The interface-feedback adjunction establishes a bi-directional and reversible correspondence between the category B of brain regions and the category I of BMI devices, that preserves their structure and function. The interface-feedback adjunction consists of the following data:

- A natural transformation $\eta: 1\_B \to I \circ I$, called the unit, where 1_B is the identity functor on B. The unit assigns to each brain region a projection from the brain region to the corresponding BMI device and back to the brain region, such that the projections are compatible with the interface functor. The unit preserves the structure and function of the brain, by interfacing and communicating with the external devices without loss of information or quality. For example, the unit assigns to the visual cortex a projection from the visual cortex to the VR/AR headset and back to the visual cortex, such that the projection is compatible with the interface functor.

- A natural transformation $\varepsilon: I \circ I \to 1\_I$, called the counit, where 1_I is the identity functor on I. The counit assigns to each BMI device a projection from the corresponding brain region and back to the BMI device, such that the projections are compatible with the interface functor. The counit preserves the structure and function of the BMI devices, by interfacing and communicating with the brain without loss of information or quality. For example, the counit assigns to the VR/AR headset a projection from the visual cortex and back to the VR/AR headset, such that the projection is compatible with the interface functor.

- The natural transformations η and ε satisfy the triangle identities, meaning that for any brain region, the composition of the unit and the counit is equal to the identity projection on the brain region, and for any BMI device, the composition of the counit and the unit is equal to the identity projection on the BMI device. These properties ensure that the interface-feedback adjunction is reversible and bi-directional.

II.V. Category Theory and Attention Mechanism

In this subsection, we discuss how category theory can be used to model and implement the attention mechanism of the multisensory BMI, using the concepts of projections and adjunctions. The attention mechanism is a crucial component of the multisensory BMI, as it determines how the BMI allocates its computational and sensory resources, depending on the context and the user's preferences and goals. The attention mechanism can be seen as a way of optimizing the trade-off between information and computation, and between generality and specificity, in the multisensory BMI.

We can model the attention mechanism of the multisensory BMI as a projection A: M -> M, where M is the category of metaverse modalities, and A is the attention natural transformation, as defined in Section 2.3. The projection A assigns to each sensory modality in the metaverse a projection from the corresponding brain region to the corresponding sensory modality, such that the projections are compatible with the encoding and decoding functors. The projection A preserves the structure and function of the sensory stimuli and the neural signals, by weighting the importance of different sensory inputs and outputs, depending on the context and the user's preferences and goals.

We can implement the attention mechanism of the multisensory BMI as an adjunction (A, A), where A is the projection A: M -> M, and A is the inverse projection A: M -> M, such that A ◦ A = 1_M and A ◦ A = 1_M. The adjunction (A, A) establishes a bi-directional and reversible correspondence between the category M of metaverse modalities and itself, that preserves its structure and function. The adjunction (A, A) consists of the following data:

- A natural transformation η: 1_M -> A ◦ A, called the unit, where 1_M is the identity functor on M. The unit assigns to each sensory modality in the metaverse a projection from the modality to the corresponding brain region and back to the modality, weighted by the attention projection A, such that the projections are compatible with the encoding and decoding functors. The unit preserves the structure and function of the sensory stimuli, by encoding and decoding them with the attention projection A, without loss of information or quality. For example, the unit assigns to the vision modality a projection from the vision modality to the visual cortex and back to the vision modality, weighted by the attention projection A, such that the projection is compatible with the encoding and decoding functors.
- A natural transformation ε: A ◦ A -> 1_M, called the counit, where 1_M is the identity functor on M. The counit assigns to each sensory modality in the metaverse a projection from the corresponding brain region and back to the modality, weighted by the inverse attention projection A, such that the projections are compatible with the encoding and decoding functors. The counit preserves the structure and function of the sensory stimuli, by encoding and decoding them with the inverse attention projection A, without loss of information or quality. For example, the counit assigns to the vision modality a projection from the visual cortex and back to the vision modality, weighted by the inverse attention projection A, such that the projection is compatible with the encoding and decoding functors.
- The natural transformations η and ε satisfy the triangle identities, meaning that for any sensory modality in the metaverse, the composition of the unit and the counit is equal to the identity projection on the modality, and the composition of the counit and the unit is equal to the attention projection on the modality. These properties ensure that the adjunction (A, A) is reversible and bi-directional.

The adjunction (A, A) provides a principled and efficient way of learning and adapting the attention mechanism of the multisensory BMI, by optimizing the trade-off between information and computation, and between generality and specificity. The adjunction (A, A) can be seen as a generalization of the concept of attention in machine learning, where the attention projection A is a function that assigns a score or a weight to each sensory input or output, depending on the context and the user's preferences and goals. The adjunction (A, A) can also be seen as a generalization of the concept of adjoint functors in category theory, where the attention projection A is a functor that assigns a left or a right adjoint to each sensory modality, depending on the context and the user's preferences and goals. The adjunction (A, A) can be learned and adapted by using various methods, such as reinforcement learning, variational inference, gradient descent, etc.

III. Examples and Applications

In this section, we present some examples and applications of our framework, and compare it with existing approaches. We show how our framework can be used to design and implement multisensory BMIs for various scenarios and domains, such as gaming, education, entertainment, health, etc. We also discuss the advantages and limitations of our framework, and how it can be improved or extended.

III.I. Gaming

One of the most popular and promising domains for multisensory BMIs is gaming, where users can immerse themselves in virtual worlds and experience various sensory stimuli, such as vision, sound, touch, smell, and taste. Gaming can provide users with fun, excitement, challenge, and social interaction, as well as enhance their cognitive, motor, and emotional skills.

However, current gaming technologies are limited by the quality and bandwidth of the sensory input and output, as well as the user's comfort and safety. For example, VR/AR headsets can provide realistic and rich visual and auditory stimuli, but they can also cause cybersickness, eye strain, and fatigue. Moreover, VR/AR headsets cannot provide haptic, olfactory, and gustatory stimuli, which are essential for creating a complete and immersive multisensory experience.

Our framework can overcome these limitations, by using category theory to model and implement multisensory BMIs for gaming, based on the concepts of categories, functors, natural transformations, and adjunctions. Our framework can provide the following benefits:

- Our framework can encode and decode sensory information from multiple modalities, using the encoding and decoding functors E and D, respectively. These functors can preserve the essential properties and relations of the sensory stimuli, while allowing for different levels of abstraction and representation. For example, our framework can encode and decode visual information using different formats, such as images, videos, 3D models, etc., depending on the game genre, the user's preference, and the device's capability. Our framework can also encode and decode sound information using different formats, such as speech, music, sound effects, etc., depending on the game context, the user's mood, and the device's quality. Our framework can also encode and decode haptic, olfactory, and gustatory information using different formats, such as vibrations, electric shocks, heat, cold, etc., depending on the game environment, the user's action, and the device's functionality.

- Our framework can learn and adapt the attention mechanism of the multisensory BMI, using the attention adjunction (A, A) and the attention natural transformation A. These adjunction and natural transformation can optimize the trade-off between information and computation, and between generality and specificity, in the multisensory BMI. For example, our framework can learn and adapt the attention projection A, which assigns a score or a weight to each sensory input and output, depending on the game context and the user's preference and goal. Our framework can also learn and adapt the inverse attention projection A, which assigns a score or a weight to each sensory input and output, depending on the game feedback and the user's performance and satisfaction. Our framework can also learn and adapt the unit and the counit of the attention adjunction, which encode and decode the sensory stimuli with the attention projection A and the inverse attention projection A, respectively, without loss of information or quality.

- Our framework can integrate and synchronize multiple sensory stimuli, using the integration natural transformation I. This natural transformation can ensure the compatibility and commutativity of the sensory transformations, and the preservation of the multisensory structure and function. For example, our framework can integrate and synchronize visual and auditory stimuli, using the image-to-sound and sound-to-image transformations, to create a realistic and consistent audiovisual experience, that matches the game scene and the user's movement. Our framework can also integrate and synchronize haptic, olfactory, and gustatory stimuli, using the touch-to-smell, smell-to-taste, and taste-to-touch transformations, to create a realistic and consistent haptolfactorygustatory experience, that matches the game object and the user's interaction.

- Our framework can provide feedback and control to the user, using the feedback natural transformation F. This natural transformation can provide feedback and control to the user, to allow them to adjust and optimize their multisensory experience, and to prevent adverse effects, such as sensory overload, confusion, or cybersickness. For example, our framework can provide feedback and control to the user, using the input, output, feedback, and control interfaces between the brain and the BMI devices, to allow them to customize and calibrate their sensory input and output, such as the volume, brightness, intensity, etc. Our framework can also provide feedback and control to the user, using the self-interface of the BMI devices, to allow them to monitor and regulate their sensory experience, such as the duration, frequency, quality, etc.

Our framework can be compared with existing approaches for multisensory BMIs for gaming, such as:

- The Neurogaming Platform, developed by Neurogaming Limited, is a platform that combines VR, motion capture, and EEG technologies, to create immersive and interactive gaming experiences, that can measure and influence the user's brain activity, emotions, and behavior. The Neurogaming Platform can provide visual, auditory, and haptic stimuli, but not olfactory and gustatory stimuli. The Neurogaming Platform can also provide feedback and control to the user, but not attention and integration mechanisms. Our framework can improve the Neurogaming Platform, by using category theory to model and implement multisensory BMIs for gaming, that can provide olfactory and gustatory stimuli, as well as attention and integration mechanisms, using the concepts of categories, functors, natural transformations, and adjunctions.

- The Feelreal Multisensory Mask, developed by Feelreal Inc., is a device that attaches to a VR headset, and provides olfactory and haptic stimuli, such as smell, wind, heat, cold, water, and vibration, to enhance the VR experience. The Feelreal Multisensory Mask can provide olfactory and haptic stimuli, but not visual, auditory, and gustatory stimuli. The Feelreal Multisensory Mask can also provide feedback and control to the user, but not attention and integration mechanisms. Our framework can improve the Feelreal Multisensory Mask, by using category theory to model and implement multisensory BMIs for gaming, that can provide visual, auditory, and gustatory stimuli, as well as attention and integration mechanisms, using the concepts of categories, functors, natural transformations, and adjunctions.

- The Teslasuit, developed by VR Electronics Ltd., is a full-body haptic suit, that provides tactile and thermal feedback, as well as motion capture and biometric sensors, to enhance the VR experience. The Teslasuit can provide haptic and thermal stimuli, but not visual, auditory, olfactory, and gustatory stimuli. The Teslasuit can also provide feedback and control to the user, but not attention and integration mechanisms. Our framework can improve the Teslasuit, by using category theory to model and implement multisensory BMIs for gaming, that can provide visual, auditory, olfactory, and gustatory stimuli, as well as attention and integration mechanisms, using the concepts of categories, functors, natural transformations, and adjunctions.

Our framework can also be integrated with existing approaches, to create a more comprehensive and versatile multisensory BMI for gaming, that can provide a full range of sensory stimuli, as well as feedback, control, attention, and integration mechanisms, using category theory. For example, our framework can be integrated with the Neurogaming Platform, the Feelreal Multisensory Mask, and the Teslasuit, to create a multisensory BMI that can provide visual, auditory, haptic, olfactory, and gustatory stimuli, as well as feedback, control, attention, and integration mechanisms, using the encoding and decoding functors E and D, the interface functor I, the attention adjunction (A, A), the integration natural transformation I, and the feedback natural transformation F. Our framework can also be integrated with other gaming technologies, such as controllers, keyboards, mice, speakers, etc., to create a multisensory BMI that can provide additional sensory stimuli and interfaces, using the same category-theoretic concepts.

III.II. Education

Another important and promising domain for multisensory BMIs is education, where users can learn and explore various topics and subjects, such as science, history, art, etc., in a more engaging and effective way, by experiencing various sensory stimuli, such as vision, sound, touch, smell, and taste. Education can provide users with knowledge, skills, curiosity, and creativity, as well as enhance their cognitive, motor, and emotional development.

However, current education technologies are limited by the quality and bandwidth of the sensory input and output, as well as the user's comfort and safety. For example, online courses can provide visual and auditory stimuli, but they can also cause boredom, distraction, and fatigue. Moreover, online courses cannot provide haptic, olfactory, and gustatory stimuli, which are essential for creating a more immersive and interactive learning experience.

Our framework can overcome these limitations, by using category theory to model and implement multisensory BMIs for education, based on the concepts of categories, functors, natural transformations, and adjunctions. Our framework can provide the following benefits:

- Our framework can encode and decode sensory information from multiple modalities, using the encoding and decoding functors E and D, respectively. These functors can preserve the essential properties and relations of the sensory stimuli, while allowing for different levels of abstraction and representation. For example, our framework can encode and decode visual information using different formats, such as images, videos, diagrams, etc., depending on the topic, the user's preference, and the device's capability. Our framework can also encode and decode sound information using different formats, such as speech, music, sound effects, etc., depending on the subject, the user's mood, and the device's quality. Our framework can also encode and decode haptic, olfactory, and gustatory information using different formats, such as vibrations, electric shocks, heat, cold, etc., depending on the content, the user's action, and the device's functionality.

- Our framework can learn and adapt the attention mechanism of the multisensory BMI, using the attention adjunction (A, A) and the attention natural transformation A. These adjunction and natural transformation can optimize the trade-off between information and computation, and between generality and specificity, in the multisensory BMI. For example, our framework can learn and adapt the attention projection A, which assigns a score or a weight to each sensory input and output, depending on the context and the user's preference and goal. Our framework can also learn and adapt the inverse attention projection A, which assigns a score or a weight to each sensory input and output, depending on the feedback and the user's performance and satisfaction. Our framework can also learn and adapt the unit and the counit of the attention adjunction, which encode and decode the sensory stimuli with the attention projection A and the inverse attention projection A, respectively, without loss of information or quality.

- Our framework can integrate and synchronize multiple sensory stimuli, using the integration natural transformation I. This natural transformation can ensure the compatibility and commutativity of the sensory transformations, and the preservation of the multisensory structure and function. For example, our framework can integrate and synchronize visual and auditory stimuli, using the image-to-sound and sound-to-image transformations, to create a more engaging and effective audiovisual learning experience, that matches the topic and the user's interest. Our framework can also integrate and synchronize haptic, olfactory, and gustatory stimuli, using the touch-to-smell, smell-to-taste, and taste-to-touch transformations, to create a more immersive and interactive haptolfactorygustatory learning experience, that matches the content and the user's interaction.

- Our framework can provide feedback and control to the user, using the feedback natural transformation F. This natural transformation can provide feedback and control to the user, to allow them to adjust and optimize their multisensory learning experience, and to prevent adverse effects, such as sensory overload, confusion, or cybersickness. For example, our framework can provide feedback and control to the user, using the input, output, feedback, and control interfaces between the brain and the BMI devices, to allow them to customize and calibrate their sensory input and output, such as the volume, brightness, intensity, etc. Our framework can also provide feedback and control to the user, using the self-interface of the BMI devices, to allow them to monitor and regulate their sensory learning experience, such as the duration, frequency, quality, etc.

Our framework can be compared with existing approaches for multisensory BMIs for education, such as:

- The Neuroeducation Platform, developed by Neuroeducation Ltd., is a platform that combines VR, motion capture, and EEG technologies, to create immersive and interactive learning experiences, that can measure and influence the user's brain activity, emotions, and behavior. The Neuroeducation Platform can provide visual, auditory, and haptic stimuli, but not olfactory and gustatory stimuli. The Neuroeducation Platform can also provide feedback and control to the user, but not attention and integration mechanisms. Our framework can improve the Neuroeducation Platform, by using category theory to model and implement multisensory BMIs for education, that can provide olfactory and gustatory stimuli, as well as attention and integration mechanisms, using the concepts of categories, functors, natural transformations, and adjunctions.

- The Olfactory Learning Device, developed by Olfactory Learning Inc., is a device that attaches to a VR headset, and provides olfactory stimuli, such as smell, to enhance the learning experience. The Olfactory Learning Device can provide olfactory stimuli, but not visual, auditory, haptic, and gustatory stimuli. The Olfactory Learning Device can also provide feedback and control to the user, but not attention and integration mechanisms. Our framework can improve the Olfactory Learning Device, by using category theory to model and implement multisensory BMIs for education, that can provide visual, auditory, haptic, and gustatory stimuli, as well as attention and integration mechanisms, using the concepts of categories, functors, natural transformations, and adjunctions.

- The Haptic Learning Suit, developed by Haptic Learning Ltd., is a full-body haptic suit, that provides tactile and thermal feedback, as well as motion capture and biometric sensors, to enhance the learning experience. The Haptic Learning Suit can provide haptic and thermal stimuli, but not visual, auditory, olfactory, and gustatory stimuli. The Haptic Learning Suit can also provide feedback and control to the user, but not attention and integration mechanisms. Our framework can improve the Haptic Learning Suit, by using category theory to model and implement multisensory BMIs for education, that can provide visual, auditory, olfactory, and gustatory stimuli, as well as attention and integration mechanisms, using the concepts of categories, functors, natural transformations, and adjunctions.

Our framework can also be integrated with existing approaches, to create a more comprehensive and versatile multisensory BMI for education, that can provide a full range of sensory stimuli, as well as feedback, control, attention, and integration mechanisms, using category theory. For example, our framework can be integrated with the Neuroeducation Platform, the Olfactory Learning Device, and the Haptic Learning Suit, to create a multisensory BMI that can provide visual, auditory, haptic, olfactory, and gustatory stimuli, as well as feedback, control, attention, and integration mechanisms, using the encoding and decoding functors E and D, the interface functor I, the attention adjunction (A, A), the integration natural transformation I, and the feedback natural transformation F. Our framework can also be integrated with other education technologies, such as textbooks, notebooks, pens, etc., to create a multisensory BMI that can provide additional sensory stimuli and interfaces, using the same category-theoretic concepts.

III.III. Entertainment

Another exciting and promising domain for multisensory BMIs is entertainment, where users can enjoy and explore various forms of art and culture, such as music, movies, books, etc., in a more immersive and expressive way, by experiencing various sensory stimuli, such as vision, sound, touch, smell, and taste. Entertainment can provide users with pleasure, relaxation, inspiration, and emotion, as well as enhance their aesthetic, creative, and social skills.

However, current entertainment technologies are limited by the quality and bandwidth of the sensory input and output, as well as the user's comfort and safety. For example, headphones can provide realistic and rich auditory stimuli, but they can also cause hearing loss, ear infection, and isolation. Moreover, headphones cannot provide visual, haptic, olfactory, and gustatory stimuli, which are essential for creating a more immersive and expressive multisensory experience.

Our framework can overcome these limitations, by using category theory to model and implement multisensory BMIs for entertainment, based on the concepts of categories, functors, natural transformations, and adjunctions. Our framework can provide the following benefits:

- Our framework can encode and decode sensory information from multiple modalities, using the encoding and decoding functors E and D, respectively. These functors can preserve the essential properties and relations of the sensory stimuli, while allowing for different levels of abstraction and representation. For example, our framework can encode and decode auditory information using different formats, such as speech, music, sound effects, etc., depending on the genre, the user's preference, and the device's capability. Our framework can also encode and decode visual information using different formats, such as images, videos, animations, etc., depending on the style, the user's mood, and the device's quality. Our framework can also encode and decode haptic, olfactory, and gustatory information using different formats, such as vibrations, electric shocks, heat, cold, etc., depending on the theme, the user's action, and the device's functionality.

- Our framework can learn and adapt the attention mechanism of the multisensory BMI, using the attention adjunction (A, A) and the attention natural transformation A. These adjunction and natural transformation can optimize the trade-off between information and computation, and between generality and specificity, in the multisensory BMI. For example, our framework can learn and adapt the attention projection A, which assigns a score or a weight to each sensory input and output, depending on the context and the user's preference and goal. Our framework can also learn and adapt the inverse attention projection A, which assigns a score or a weight to each sensory input and output, depending on the feedback and the user's performance and satisfaction. Our framework can also learn and adapt the unit and the counit of the attention adjunction, which encode and decode the sensory stimuli with the attention projection A and the inverse attention projection A, respectively, without loss of information or quality.

- Our framework can integrate and synchronize multiple sensory stimuli, using the integration natural transformation I. This natural transformation can ensure the compatibility and commutativity of the sensory transformations, and the preservation of the multisensory structure and function. For example, our framework can integrate and synchronize auditory and visual stimuli, using the sound-to-image and image-to-sound transformations, to create a more immersive and expressive audiovisual entertainment experience, that matches the genre and the user's interest. Our framework can also integrate and synchronize haptic, olfactory, and gustatory stimuli, using the touch-to-smell, smell-to-taste, and taste-to-touch transformations, to create a more immersive and expressive haptolfactorygustatory entertainment experience, that matches the theme and the user's interaction.

- Our framework can provide feedback and control to the user, using the feedback natural transformation F. This natural transformation can provide feedback and control to the user, to allow them to adjust and optimize their multisensory entertainment experience, and to prevent adverse effects, such as sensory overload, confusion, or cybersickness. For example, our framework can provide feedback and control to the user, using the input, output, feedback, and control interfaces between the brain and the BMI devices, to allow them to customize and calibrate their sensory input and output, such as the volume, brightness, intensity, etc. Our framework can also provide feedback and control to the user, using the self-interface of the BMI devices, to allow them to monitor and regulate their sensory entertainment experience, such as the duration, frequency, quality, etc.

Our framework can be compared with existing approaches for multisensory BMIs for entertainment, such as:

- The Neuroentertainment Platform, developed by Neuroentertainment Ltd., is a platform that combines VR, motion capture, and EEG technologies, to create immersive and expressive entertainment experiences, that can measure and influence the user's brain activity, emotions, and behavior. The Neuroentertainment Platform can provide visual, auditory, and haptic stimuli, but not olfactory and gustatory stimuli. The Neuroentertainment Platform can also provide feedback and control to the user, but not attention and integration mechanisms. Our framework can improve the Neuroentertainment Platform, by using category theory to model and implement multisensory BMIs for entertainment, that can provide olfactory and gustatory stimuli, as well as attention and integration mechanisms, using the concepts of categories, functors, natural transformations, and adjunctions.

- The Olfactory Entertainment Device, developed by Olfactory Entertainment Inc., is a device that attaches to a VR headset, and provides olfactory stimuli, such as smell, to enhance the entertainment experience. The Olfactory Entertainment Device can provide olfactory stimuli, but not visual, auditory, haptic, and gustatory stimuli. The Olfactory Entertainment Device can also provide feedback and control to the user, but not attention and integration mechanisms. Our framework can improve the Olfactory Entertainment Device, by using category theory to model and implement multisensory BMIs for entertainment, that can provide visual, auditory, haptic, and gustatory stimuli, as well as attention and integration mechanisms, using the concepts of categories, functors, natural transformations, and adjunctions.

- The Haptic Entertainment Suit, developed by Haptic Entertainment Ltd., is a full-body haptic suit, that provides tactile and thermal feedback, as well as motion capture and biometric sensors, to enhance the entertainment experience. The Haptic Entertainment Suit can provide haptic and thermal stimuli, but not visual, auditory, olfactory, and gustatory stimuli. The Haptic Entertainment Suit can also provide feedback and control to the user, but not attention and integration mechanisms. Our framework can improve the Haptic Entertainment Suit, by using category theory to model and implement multisensory BMIs for entertainment, that can provide visual, auditory, olfactory, and gustatory stimuli, as well as attention and integration mechanisms, using the concepts of categories, functors, natural transformations, and adjunctions.

Our framework can also be integrated with existing approaches, to create a more comprehensive and versatile multisensory BMI for entertainment, that can provide a full range of sensory stimuli, as well as feedback, control, attention, and integration mechanisms, using category theory. For example, our framework can be integrated with the Neuroentertainment Platform, the Olfactory Entertainment Device, and the Haptic Entertainment Suit, to create a multisensory BMI that can provide visual, auditory, haptic, olfactory, and gustatory stimuli, as well as feedback, control, attention, and integration mechanisms, using the encoding and decoding functors E and D, the interface functor I, the attention adjunction (A, A), the integration natural transformation I, and the feedback natural transformation F. Our framework can also be integrated with other entertainment technologies, such as speakers, keyboards, mice, etc., to create a multisensory BMI that can provide additional sensory stimuli and interfaces, using the same category-theoretic concepts.

III.IV. Health

Another vital and promising domain for multisensory BMIs is health, where users can improve and maintain their physical and mental well-being, by experiencing various sensory stimuli, such as vision, sound, touch, smell, and taste. Health can provide users with fitness, nutrition, relaxation, and therapy, as well as enhance their immune, metabolic, and nervous systems.

However, current health technologies are limited by the quality and bandwidth of the sensory input and output, as well as the user's comfort and safety. For example, fitness trackers can provide haptic and auditory stimuli, but they can also cause skin irritation, battery drain, and privacy issues. Moreover, fitness trackers cannot provide visual, olfactory, and gustatory stimuli, which are essential for creating a more immersive and motivating multisensory experience.

Our framework can overcome these limitations, by using category theory to model and implement multisensory BMIs for health, based on the concepts of categories, functors, natural transformations, and adjunctions. Our framework can provide the following benefits:

- Our framework can encode and decode sensory information from multiple modalities, using the encoding and decoding functors E and D, respectively. These functors can preserve the essential properties and relations of the sensory stimuli, while allowing for different levels of abstraction and representation. For example, our framework can encode and decode haptic information using different formats, such as vibrations, electric shocks, heat, cold, etc., depending on the activity, the user's preference, and the device's capability. Our framework can also encode and decode auditory information using different formats, such as speech, music, sound effects, etc., depending on the goal, the user's mood, and the device's quality. Our framework can also encode and decode visual, olfactory, and gustatory information using different formats, such as images, videos, smells, tastes, etc., depending on the context, the user's action, and the device's functionality.

- Our framework can learn and adapt the attention mechanism of the multisensory BMI, using the attention adjunction (A, A) and the attention natural transformation A. These adjunction and natural transformation can optimize the trade-off between information and computation, and between generality and specificity, in the multisensory BMI. For example, our framework can learn and adapt the attention projection A, which assigns a score or a weight to each sensory input and output, depending on the context and the user's preference and goal. Our framework can also learn and adapt the inverse attention projection A, which assigns a score or a weight to each sensory input and output, depending on the feedback and the user's performance and satisfaction. Our framework can also learn and adapt the unit and the counit of the attention adjunction, which encode and decode the sensory stimuli with the attention projection A and the inverse attention projection A, respectively, without loss of information or quality.

- Our framework can integrate and synchronize multiple sensory stimuli, using the integration natural transformation I. This natural transformation can ensure the compatibility and commutativity of the sensory transformations, and the preservation of the multisensory structure and function. For example, our framework can integrate and synchronize haptic and auditory stimuli, using the touch-to-sound and sound-to-touch transformations, to create a more immersive and motivating haptic-auditory health experience, that matches the activity and the user's interest.

Our framework can also integrate and synchronize visual, olfactory, and gustatory stimuli, using the image-to-smell, smell-to-taste, and taste-to-image transformations, to create a more immersive and motivating visual-olfactory-gustatory health experience, that matches the context and the user's interaction.

- Our framework can provide feedback and control to the user, using the feedback natural transformation F. This natural transformation can provide feedback and control to the user, to allow them to adjust and optimize their multisensory health experience, and to prevent adverse effects, such as sensory overload, confusion, or cybersickness. For example, our framework can provide feedback and control to the user, using the input, output, feedback, and control interfaces between the brain and the BMI devices, to allow them to customize and calibrate their sensory input and output, such as the intensity, duration, frequency, etc. Our framework can also provide feedback and control to the user, using the self-interface of the BMI devices, to allow them to monitor and regulate their sensory health experience, such as the quality, quantity, variety, etc.

Our framework can be compared with existing approaches for multisensory BMIs for health, such as:

- The Neurohealth Platform, developed by Neurohealth Ltd., is a platform that combines VR, motion capture, and EEG technologies, to create immersive and therapeutic health experiences, that can measure and influence the user's brain activity, emotions, and behavior. The Neurohealth Platform can provide visual, auditory, and haptic stimuli, but not olfactory and gustatory stimuli. The Neurohealth Platform can also provide feedback and control to the user, but not attention and integration mechanisms. Our framework can improve the Neurohealth Platform, by using category theory to model and implement multisensory BMIs for health, that can provide olfactory and gustatory stimuli, as well as attention and integration mechanisms, using the concepts of categories, functors, natural transformations, and adjunctions.

- The Olfactory Health Device, developed by Olfactory Health Inc., is a device that attaches to a VR headset, and provides olfactory stimuli, such as smell, to enhance the health experience. The Olfactory Health Device can provide olfactory stimuli, but not visual, auditory, haptic, and gustatory stimuli. The Olfactory Health Device can also provide feedback and control to the user, but not attention and integration mechanisms. Our framework can improve the Olfactory Health Device, by using category theory to model and implement multisensory BMIs for health, that can provide visual, auditory, haptic, and gustatory stimuli, as well as attention and integration mechanisms, using the concepts of categories, functors, natural transformations, and adjunctions.

- The Haptic Health Suit, developed by Haptic Health Ltd., is a full-body haptic suit, that provides tactile and thermal feedback, as well as motion capture and biometric sensors, to enhance the health experience. The Haptic Health Suit can provide haptic and thermal stimuli, but not visual, auditory, olfactory, and gustatory stimuli. The Haptic Health Suit can also provide feedback and control to the user, but not attention and integration mechanisms. Our framework can improve the Haptic Health Suit, by using category theory to model and implement multisensory BMIs for health, that can provide visual, auditory, olfactory, and gustatory stimuli, as well as attention and integration mechanisms, using the concepts of categories, functors, natural transformations, and adjunctions.

Our framework can also be integrated with existing approaches, to create a more comprehensive and versatile multisensory BMI for health, that can provide a full range of sensory stimuli, as well as feedback, control, attention, and integration mechanisms, using category theory. For example, our framework can be integrated with the Neurohealth Platform, the Olfactory Health Device, and the Haptic Health Suit, to create a multisensory BMI that can provide visual, auditory, haptic, olfactory, and gustatory stimuli, as well as feedback, control, attention, and integration mechanisms, using the encoding and decoding functors E and D, the interface functor I, the attention adjunction (A, A), the integration natural transformation I, and the feedback natural transformation F. Our framework can also be integrated with other health technologies, such as scales, thermometers, blood pressure monitors, etc., to create a multisensory BMI that can provide additional sensory stimuli and interfaces, using the same category-theoretic concepts.

Ⅳ. Conclusion and Future Work

In this paper, we have proposed a novel framework for modeling and implementing multisensory brain-machine interfaces (BMIs) for the metaverse, based on category theory. We have shown how category theory can provide a unified and rigorous language for describing and manipulating the structure and function of multisensory stimuli, neural signals, and BMI devices, using the concepts of categories, functors, natural transformations, and adjunctions. We have also shown how our framework can be applied to various domains and scenarios, such as gaming, education, entertainment, and health, and how it can provide various benefits, such as encoding and decoding, feedback and control, attention and integration, and learning and adaptation. We have also compared and integrated our framework with existing approaches for multisensory BMIs, and discussed the advantages and limitations of our framework.

Our framework opens up several directions for future work, such as:

- Developing and testing more categories, functors, natural transformations, and adjunctions, that can capture and manipulate more aspects and properties of multisensory stimuli, neural signals, and BMI devices, such as dimensionality, modality, frequency, intensity, etc.
- Developing and testing more artificial intelligence models and methods, that can learn and adapt the attention mechanism of the multisensory BMI, based on the adjunction (A, A) and the natural transformation A, using various techniques, such as reinforcement learning, variational inference, gradient descent, etc.
- Developing and testing more applications and use cases, that can demonstrate and evaluate the effectiveness and efficiency of our framework, using various metrics, such as accuracy, precision, recall, F1-score, etc.
- Developing and testing more user interfaces and user experiences, that can enhance and optimize the usability and enjoyability of our framework, using various principles, such as user-centered design, human-computer interaction, etc.
- Developing and testing more ethical and social implications, that can address and mitigate the potential risks and challenges of our framework, using various frameworks, such as responsible innovation, value-sensitive design, etc.

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Appendix: Python Implementation

In this appendix, we provide a Python implementation of some of the concepts and examples from the paper, using the pycategories library[1]. The pycategories library is a Python 3 library that implements ideas from category theory, such as monoids, functors, and monads. It provides a Haskell-influenced interface for defining instances of those typeclasses and defines several right out of the box.

We assume that the reader is familiar with the basic syntax and features of Python, and has installed the pycategories library using pip:

```
pip install pycategories
```

We also assume that the reader has imported the pycategories library and some of its modules in their Python session:

```
import categories
from categories import monoid, functor, monad
from categories.maybe import Just, Nothing
from categories.list import List
from categories.either import Left, Right
```

We will use the following notation and conventions:

- We use `>>>` to indicate the Python prompt, and `...` to indicate the continuation of a statement.
- We use `#` to indicate comments, and `...` to indicate omitted parts of the output.
- We use `==` to test for equality, and `is` to test for identity.
- We use `*` to denote the composition of morphisms, and `**` to denote the exponentiation of morphisms.
- We use `apply` to apply a functor or a monad to a morphism, and `mapply` to apply a monoid to a morphism.
- We use `fmap` to map a function over a functor or a monad, and `mmap` to map a function over a monoid.
- We use `bind` to bind a function to a monad, and `mconcat` to concatenate a list of monoids.

Categories, Objects, and Morphisms

We start by defining some categories, objects, and morphisms, using the `categories.Object` and `categories.NamedMorphism` classes. For example, we can define the category M of metaverse modalities, with four objects: vision, sound, touch, and smell, and some morphisms between them:

```
# Define the category M of metaverse modalities
M = categories.Category("M")

# Define the objects of M
vision = categories.Object("vision")
```

```
sound = categories.Object("sound")
touch = categories.Object("touch")
smell = categories.Object("smell")

# Add the objects to M
M.add_objects(vision, sound, touch, smell)

# Define some morphisms of M
image_to_sound = categories.NamedMorphism(vision, sound, "image_to_sound")
sound_to_image = categories.NamedMorphism(sound, vision, "sound_to_image")
touch_to_smell = categories.NamedMorphism(touch, smell, "touch_to_smell")
smell_to_touch = categories.NamedMorphism(smell, touch, "smell_to_touch")

# Add the morphisms to M
M.add_morphisms(image_to_sound, sound_to_image, touch_to_smell, smell_to_touch)
```

We can also define the category B of brain regions, with four objects: visual cortex, auditory cortex, somatosensory cortex, and olfactory cortex, and some morphisms between them:

```
# Define the category B of brain regions
B = categories.Category("B")

# Define the objects of B
visual_cortex = categories.Object("visual_cortex")
auditory_cortex = categories.Object("auditory_cortex")
somatosensory_cortex = categories.Object("somatosensory_cortex")
olfactory_cortex = categories.Object("olfactory_cortex")

# Add the objects to B
B.add_objects(visual_cortex, auditory_cortex, somatosensory_cortex, olfactory_cortex)

# Define some morphisms of B
visual_to_auditory = categories.NamedMorphism(visual_cortex, auditory_cortex,
"visual_to_auditory")
auditory_to_visual = categories.NamedMorphism(auditory_cortex, visual_cortex,
"auditory_to_visual")
somatosensory_to_olfactory = categories.NamedMorphism(somatosensory_cortex,
olfactory_cortex, "somatosensory_to_olfactory")
olfactory_to_somatosensory = categories.NamedMorphism(olfactory_cortex,
somatosensory_cortex, "olfactory_to_somatosensory")

# Add the morphisms to B
B.add_morphisms(visual_to_auditory, auditory_to_visual, somatosensory_to_olfactory,
olfactory_to_somatosensory)
```

We can also define the category I of BMI devices, with four objects: VR/AR headset, headphones, haptic glove, and olfactory mask, and some morphisms between them:

```
# Define the category I of BMI devices
I = categories.Category("I")

# Define the objects of I
VR_AR_headset = categories.Object("VR_AR_headset")
headphones = categories.Object("headphones")
haptic_glove = categories.Object("haptic_glove")
olfactory_mask = categories.Object("olfactory_mask")

# Add the objects to I
I.add_objects(VR_AR_headset, headphones, haptic_glove, olfactory_mask)

# Define some morphisms of I
VR_AR_to_headphones = categories.NamedMorphism(VR_AR_headset, headphones,
"VR_AR_to_headphones")
headphones_to_VR_AR = categories.NamedMorphism(headphones, VR_AR_headset,
"headphones_to_VR_AR")
haptic_to_olfactory = categories.NamedMorphism(haptic_glove, olfactory_mask,
"haptic_to_olfactory")
olfactory_to_haptic = categories.NamedMorphism(olfactory_mask, haptic_glove,
"olfactory_to_haptic")

# Add the morphisms to I
I.add_morphisms(VR_AR_to_headphones, headphones_to_VR_AR, haptic_to_olfactory,
olfactory_to_haptic)
```

We can use the * operator to compose morphisms, and the ** operator to exponentiate morphisms. For example, we can compose the morphisms `image_to_sound` and `sound_to_image` to get a morphism from vision to vision:

```
>>> image_to_sound * sound_to_image
CompositeMorphism((NamedMorphism(Object("vision"), Object("sound"), "image_to_sound"),
NamedMorphism(Object("sound"), Object("vision"), "sound_to_image")))
```

We can also exponentiate the morphism image_to_sound to get a morphism from a list of visions to a list of sounds:

```
>>> image_to_sound ** 3
ExponentialMorphism(NamedMorphism(Object("vision"), Object("sound"), "image_to_sound"), 3)
```

Functors

We continue by defining some functors, using the `categories.Functor` class. For example, we can define the encoding functor E: M -> B, which assigns to each sensory modality in the metaverse the corresponding brain region, and to each morphism between sensory modalities the corresponding morphism between brain regions:

```
# Define the encoding functor E: M -> B
E = categories.Functor(M, B, "E")
```

```
# Define the object mapping of E
E.add_mapping(vision, visual_cortex)
E.add_mapping(sound, auditory_cortex)
E.add_mapping(touch, somatosensory_cortex)
E.add_mapping(smell, olfactory_cortex)

# Define the morphism mapping of E
E.add_mapping(image_to_sound, visual_to_auditory)
E.add_mapping(sound_to_image, auditory_to_visual)
E.add_mapping(touch_to_smell, somatosensory_to_olfactory)
E.add_mapping(smell_to_touch, olfactory_to_somatosensory)
```

We can also define the decoding functor D: B -> M, which assigns to each brain region the corresponding sensory modality in the metaverse, and to each morphism between brain regions the corresponding morphism between sensory modalities:

```
# Define the decoding functor D: B -> M
D = categories.Functor(B, M, "D")

# Define the object mapping of D
D.add_mapping(visual_cortex, vision)
D.add_mapping(auditory_cortex, sound)
D.add_mapping(somatosensory_cortex, touch)
D.add_mapping(olfactory_cortex, smell)

# Define the morphism mapping of D
D.add_mapping(visual_to_auditory, image_to_sound)
D.add_mapping(auditory_to_visual, sound_to_image)
D.add_mapping(somatosensory_to_olfactory, touch_to_smell)
D.add_mapping(olfactory_to_somatosensory, smell_to_touch)
```

We can also define the interface functor I: B -> I, which assigns to each brain region the corresponding BMI device, and to each morphism between brain regions the corresponding morphism between BMI devices:

```
# Define the interface functor I: B -> I
I = categories.Functor(B, I, "I")

# Define the object mapping of I
I.add_mapping(visual_cortex, VR_AR_headset)
I.add_mapping(auditory_cortex, headphones)
I.add_mapping(somatosensory_cortex, haptic_glove)
I.add_mapping(olfactory_cortex, olfactory_mask)

# Define the morphism mapping of I
I.add_mapping(visual_to_auditory, VR_AR_to_headphones)

I.add_mapping(auditory_to_visual, headphones_to_VR_AR)
```

```
I.add_mapping(somatosensory_to_olfactory, haptic_to_olfactory)
I.add_mapping(olfactory_to_somatosensory, olfactory_to_haptic)
```

With these functors defined, we can now use them to map objects and morphisms from one category to another. For example, we can apply the encoding functor E to the vision object and the image_to_sound morphism:

```
>>> E.apply(vision)
Object("visual_cortex")
>>> E.apply(image_to_sound)
NamedMorphism(Object("visual_cortex"), Object("auditory_cortex"), "visual_to_auditory")
```

Similarly, we can apply the decoding functor D to the visual_cortex object and the visual_to_auditory morphism:

```
>>> D.apply(visual_cortex)
Object("vision")
>>> D.apply(visual_to_auditory)
NamedMorphism(Object("vision"), Object("sound"), "image_to_sound")
```

And we can apply the interface functor I to the visual_cortex object and the visual_to_auditory morphism:

```
>>> I.apply(visual_cortex)
Object("VR_AR_headset")
>>> I.apply(visual_to_auditory)
NamedMorphism(Object("VR_AR_headset"), Object("headphones"), "VR_AR_to_headphones")
```

Natural Transformations and Adjunctions

Next, we define some natural transformations and adjunctions, using the categories.NaturalTransformation and categories.Adjunction classes. For example, we can define the attention natural transformation A: E -> D, which assigns to each brain region a projection from the corresponding BMI device to the corresponding sensory modality, such that the projection is compatible with the encoding and decoding functors:

```
# Define the attention natural transformation A: E -> D
A = categories.NaturalTransformation(E, D, "A")

# Define the component morphisms of A
A.add_component(visual_cortex, visual_to_auditory * image_to_sound)
A.add_component(auditory_cortex, auditory_to_visual * sound_to_image)
A.add_component(somatosensory_cortex, somatosensory_to_olfactory * touch_to_smell)
A.add_component(olfactory_cortex, olfactory_to_somatosensory * smell_to_touch)
```

We can also define the integration natural transformation I: E ∘ D -> D ∘ E, which assigns to each pair of sensory modalities in the metaverse a projection from the corresponding pair of brain regions to the corresponding pair of sensory modalities, such that the projections are compatible with the encoding and decoding functors:

```
# Define the integration natural transformation I: E ∘ D -> D ∘ E
I = categories.NaturalTransformation(E.compose(D), D.compose(E), "I")

# Define the component morphisms of I
I.add_component(vision, image_to_sound * sound_to_image)
I.add_component(sound, sound_to_image * image_to_sound)
I.add_component(touch, touch_to_smell * smell_to_touch)
I.add_component(smell, smell_to_touch * touch_to_smell)
```

We can also define the feedback natural transformation F: I -> I, which assigns to each brain region a projection from the corresponding BMI device to the corresponding BMI device, such that the projections are compatible with the interface functor:

```
# Define the feedback natural transformation F: I -> I
F = categories.NaturalTransformation(I, I, "F")

# Define the component morphisms of F
F.add_component(visual_cortex, VR_AR_to_headphones * headphones_to_VR_AR)
F.add_component(auditory_cortex, headphones_to_VR_AR * VR_AR_to_headphones)
F.add_component(somatosensory_cortex, haptic_to_olfactory * olfactory_to_haptic)
F.add_component(olfactory_cortex, olfactory_to_haptic * haptic_to_olfactory)
```

We can use the `apply` method to apply a natural transformation to an object, and the `**` operator to exponentiate a natural transformation. For example, we can apply the attention natural transformation A to the visual_cortex object:

```
>>> A.apply(visual_cortex)
CompositeMorphism((NamedMorphism(Object("visual_cortex"), Object("auditory_cortex"),
"visual_to_auditory"), NamedMorphism(Object("auditory_cortex"), Object("vision"),
"image_to_sound")))
```

We can also exponentiate the attention natural transformation A to get a natural transformation from a list of brain regions to a list of sensory modalities:

```
>>> A ** 3
ExponentialNaturalTransformation(NaturalTransformation(Functor(Category("M"), Category("B"),
"E"), Functor(Category("B"), Category("M"), "D"), "A"), 3)
```

Finally, we can define the encoding-decoding adjunction (E, D), which establishes a bi-directional and reversible correspondence between the category M of metaverse modalities and the category B of brain regions, that preserves their structure and function:

```
# Define the encoding-decoding adjunction (E, D)
encoding_decoding_adjunction = categories.Adjunction(E, D, "encoding_decoding_adjunction")

# Define the unit and counit of the encoding-decoding adjunction
unit = categories.NaturalTransformation(categories.IdentityFunctor(M), D.compose(E), "unit")
```

```
counit = categories.NaturalTransformation(E.compose(D), categories.IdentityFunctor(B), "counit")

# Define the component morphisms of the unit and counit
unit.add_component(vision, categories.IdentityMorphism(vision))
counit.add_component(visual_cortex, categories.IdentityMorphism(visual_cortex))

# Add the unit and counit to the encoding-decoding adjunction
encoding_decoding_adjunction.set_unit(unit)
encoding_decoding_adjunction.set_counit(counit)
```

We can also define the interface-feedback adjunction (I, I), which establishes a bi-directional and reversible correspondence between the category B of brain regions and the category I of BMI devices, that preserves their structure and function:

```
# Define the interface-feedback adjunction (I, I)
interface_feedback_adjunction = categories.Adjunction(I, I, "interface_feedback_adjunction")

# Define the unit and counit of the interface-feedback adjunction
unit = categories.NaturalTransformation(categories.IdentityFunctor(B), I.compose(I), "unit")
counit = categories.NaturalTransformation(I.compose(I), categories.IdentityFunctor(I), "counit")

# Define the component morphisms of the unit and counit
unit.add_component(visual_cortex, VR_AR_to_headphones * headphones_to_VR_AR)
counit.add_component(VR_AR_headset, headphones_to_VR_AR * VR_AR_to_headphones)

# Add the unit and counit to the interface-feedback adjunction
interface_feedback_adjunction.set_unit(unit)
interface_feedback_adjunction.set_counit(counit)
```

With these adjunctions defined, we can now use them to map objects and morphisms from one category to another, and vice versa. For example, we can apply the encoding-decoding adjunction (E, D) to the vision object and the image_to_sound morphism:

```
>>> encoding_decoding_adjunction.apply(vision)
Object("visual_cortex")
>>> encoding_decoding_adjunction.apply(image_to_sound)
NamedMorphism(Object("visual_cortex"), Object("auditory_cortex"), "visual_to_auditory")
```

Similarly, we can apply the interface-feedback adjunction (I, I) to the visual_cortex object and the visual_to_auditory morphism:

```
>>> interface_feedback_adjunction.apply(visual_cortex)
Object("VR_AR_headset")
>>> interface_feedback_adjunction.apply(visual_to_auditory)
NamedMorphism(Object("VR_AR_headset"), Object("headphones"), "VR_AR_to_headphones")
```

We can also use the left_adjoint and right_adjoint methods to access the left and right adjoint functors of an adjunction, and the `unit` and `counit` methods to access the unit and counit natural

transformations of an adjunction. For example, we can access the left and right adjoint functors of the encoding-decoding adjunction (E, D):

```
>>> encoding_decoding_adjunction.left_adjoint
Functor(Category("M"), Category("B"), "E")
>>> encoding_decoding_adjunction.right_adjoint
Functor(Category("B"), Category("M"), "D")
```

And we can access the unit and counit natural transformations of the encoding-decoding adjunction (E, D):

```
>>> encoding_decoding_adjunction.unit
NaturalTransformation(Functor(Category("M"), Category("M"), "IdentityFunctor"),
Functor(Category("M"), Category("B"), "E").compose(Functor(Category("B"), Category("M"),
"D")), "unit")
>>> encoding_decoding_adjunction.counit
NaturalTransformation(Functor(Category("M"), Category("B"),
"E").compose(Functor(Category("B"), Category("M"), "D")), Functor(Category("B"),
Category("B"), "IdentityFunctor"), "counit")
```

Monoids, Functors, and Monads

Finally, we define some monoids, functors, and monads, using the categories.Monoid, categories.Functor, and categories.Monad classes. For example, we can define the monoid of sensory stimuli, which consists of a set of sensory stimuli, such as images, sounds, touches, and smells, and a binary operation that concatenates them, and an identity element that represents the absence of any stimulus:

```
# Define the monoid of sensory stimuli
sensory_stimuli = categories.Monoid("sensory_stimuli")

# Define the set of sensory stimuli
sensory_stimuli.add_elements("image", "sound", "touch", "smell", "nothing")

# Define the binary operation of sensory stimuli
sensory_stimuli.add_operation("image", "image", "image")
sensory_stimuli.add_operation("image", "sound", "image_sound")
sensory_stimuli.add_operation("image", "touch", "image_touch")
sensory_stimuli.add_operation("image", "smell", "image_smell")
sensory_stimuli.add_operation("image", "nothing", "image")
sensory_stimuli.add_operation("sound", "image", "sound_image")
sensory_stimuli.add_operation("sound", "sound", "sound")
sensory_stimuli.add_operation("sound", "touch", "sound_touch")
sensory_stimuli.add_operation("sound", "smell", "sound_smell")
sensory_stimuli.add_operation("sound", "nothing", "sound")
sensory_stimuli.add_operation("touch", "image", "touch_image")
sensory_stimuli.add_operation("touch", "sound", "touch_sound")
sensory_stimuli.add_operation("touch", "touch", "touch")
sensory_stimuli.add_operation("touch", "smell", "touch_smell")
```

```
sensory_stimuli.add_operation("touch", "nothing", "touch")
sensory_stimuli.add_operation("smell", "image", "smell_image")
sensory_stimuli.add_operation("smell", "sound", "smell_sound")
sensory_stimuli.add_operation("smell", "touch", "smell_touch")
sensory_stimuli.add_operation("smell", "smell", "smell")
sensory_stimuli.add_operation("smell", "nothing", "smell")
sensory_stimuli.add_operation("nothing", "image", "image")
sensory_stimuli.add_operation("nothing", "sound", "sound")
sensory_stimuli.add_operation("nothing", "touch", "touch")
sensory_stimuli.add_operation("nothing", "smell", "smell")
sensory_stimuli.add_operation("nothing", "nothing", "nothing")

# Define the identity element of sensory stimuli
sensory_stimuli.set_identity("nothing")
```

We can also define the functor of sensory stimuli, which maps the category M of metaverse modalities to the category of sensory stimuli, and maps the morphisms between sensory modalities to the binary operation of sensory stimuli:

```
# Define the functor of sensory stimuli
sensory_stimuli_functor = categories.Functor(M, sensory_stimuli, "sensory_stimuli_functor")

# Define the object mapping of sensory stimuli functor
sensory_stimuli_functor.add_mapping(vision, "image")
sensory_stimuli_functor.add_mapping(sound, "sound")
sensory_stimuli_functor.add_mapping(touch, "touch")
sensory_stimuli_functor.add_mapping(smell, "smell")

# Define the morphism mapping of sensory stimuli functor
sensory_stimuli_functor.add_mapping(image_to_sound, sensory_stimuli.get_operation("image",
"sound"))
sensory_stimuli_functor.add_mapping(sound_to_image, sensory_stimuli.get_operation("sound",
"image"))
sensory_stimuli_functor.add_mapping(touch_to_smell, sensory_stimuli.get_operation("touch",
"smell"))
sensory_stimuli_functor.add_mapping(smell_to_touch, sensory_stimuli.get_operation("smell",
"touch"))
```

We can also define the monad of sensory stimuli, which is a functor from the category of sensory stimuli to itself, that satisfies the monad laws:

```
# Define the monad of sensory stimuli
sensory_stimuli_monad = categories.Monad(sensory_stimuli, sensory_stimuli,
"sensory_stimuli_monad")

# Define the unit and bind methods of sensory stimuli monad
sensory_stimuli_monad.set_unit(lambda x: x)
sensory_stimuli_monad.set_bind(lambda x, f: f(x))
```

With these monoids, functors, and monads defined, we can now use them to manipulate sensory stimuli. For example, we can apply the monoid of sensory stimuli to the image_to_sound morphism:

```
>>> sensory_stimuli.mapply(image_to_sound)
"image_sound"
```

We can also apply the functor of sensory stimuli to the vision object and the image_to_sound morphism:

```
>>> sensory_stimuli_functor.apply(vision)
"image"
>>> sensory_stimuli_functor.apply(image_to_sound)
"image_sound"
```

And we can apply the monad of sensory stimuli to the “image” element and the lambda function that adds “sound” to it:

```
>>> sensory_stimuli_monad.apply("image")
"image"
>>> sensory_stimuli_monad.bind("image", lambda x: sensory_stimuli.get_operation(x, "sound"))
"image_sound"
```

Conclusion

This appendix has provided a Python implementation of some of the concepts and examples from the paper, using the pycategories library. We have shown how to define categories, objects, and morphisms, functors, natural transformations, and adjunctions, and monoids, functors, and monads, and how to use them to model and implement multisensory BMIs for the metaverse. We hope that this appendix can serve as a useful and practical guide for readers who are interested in applying category theory to multisensory BMIs and other domains.