FIBERWISE COFINAL FUNCTORS AND FUBINI FORMULAE FOR POINTWISE KAN EXTENSIONS

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Abstract. We isolate a simple finality condition that explains when a functor $p : E \to B$ permits pointwise left Kan extensions to be computed strictly fiber by strictly fiber. For each $b \in B$, let E_b denote the strict fiber of p over b, and let

$$\iota_b : E_b \longrightarrow (p \downarrow b)$$

be the fully faithful functor sending $e \in E_b$ to $(e, 1_b)$. We call p *left fiberwise cofinal* if every ι_b is final. The main theorem proves that this condition is equivalent to the formula

$$(\mathrm{Lan}_p F)(b) \cong \operatorname*{colim}_{e \in E_b} F(e)$$

for every set-small diagram $F : E \to C$ with C cocomplete. Consequently, for every $K : B \to D$,

$$\mathrm{Lan}_{Kp} F \cong \mathrm{Lan}_K (p_! F), \qquad (p_! F)(b) = \operatorname*{colim}_{e \in E_b} F(e).$$

Grothendieck opfibrations are left fiberwise cofinal, giving the usual Fubini formula for Grothendieck constructions as a corollary. The condition is strictly weaker than being an opfibration; an explicit non-opfibrational example is given. The dual theory for right Kan extensions is also stated.

1. Introduction

Let

$$p : E \to B$$

be a functor. The pointwise formula for left Kan extensions says that

$$(\mathrm{Lan}_p F)(b) \cong \operatorname*{colim}_{(e, \beta : p(e) \to b) \in (p \downarrow b)} F(e).$$

In general this colimit is indexed not by the strict fiber E_b, but by the whole comma category $(p \downarrow b)$. The question studied here is:

> When can the comma category $(p \downarrow b)$ be replaced by the strict fiber E_b, for all diagrams $F : E \to C$?

The answer is exactly the finality of the canonical functor

$$\iota_b : E_b \to (p \downarrow b), \qquad e \longmapsto (e, 1_b).$$

This note calls such functors *left fiberwise cofinal*. The terminology is chosen to avoid the overloaded word “exact”, since no exact-category or abelian-category structure is involved. The condition says that every object of $(p \downarrow b)$ is cofinally approximated by objects lying strictly over b.

The main theorem proves the equivalence of the following three statements:

(i) each $\iota_b : E_b \to (p \downarrow b)$ is final;

(ii) for every cocomplete C and every $F : E \to C$,

$$(\mathrm{Lan}_p F)(b) \cong \operatorname*{colim}_{E_b} F;$$

(iii) for every $K : B \to D$,

$$\mathrm{Lan}_{Kp} F \cong \mathrm{Lan}_K(p_! F), \qquad (p_! F)(b) = \operatorname*{colim}_{E_b} F.$$

Grothendieck opfibrations satisfy this condition, but the condition is weaker. Section 6 gives a concrete example of a left fiberwise cofinal functor that is not an opfibration.

All categories in this note are assumed small unless otherwise stated. Target categories C are locally small and cocomplete, or complete in the dual results. These hypotheses ensure that the relevant comma categories and colimits in the pointwise Kan extension formula exist.

The results are elementary and should be viewed as an explicit ordinary-categorical formulation of standard facts about final functors, pointwise Kan extensions, and fibrational decompositions of colimits; see, for example, Mac Lane [1], Kelly [2], Riehl [3], Loregian [4], Gray [5], Street [6], Hermida [7], Peschke–Tholen [8], and Cisinski [9].

2. Strict fibers and comma categories

Let

$$p : E \to B$$

be a functor. For $b \in B$, the *strict fiber* E_b is the subcategory of E whose objects are those $e \in E$ with

$$p(e) = b,$$

and whose morphisms are those $\alpha : e \to e'$ with

$$p(\alpha) = 1_b.$$

The comma category

$$(p \downarrow b)$$

has objects pairs

$$(e, \beta : p(e) \to b).$$

A morphism

$$(e, \beta) \to (e', \beta')$$

is a morphism $\alpha : e \to e'$ in E such that

$$\beta' p(\alpha) = \beta.$$

There is a fully faithful functor

$$\iota_b : E_b \to (p \downarrow b)$$

defined by

$$e \longmapsto (e, 1_b),$$

and on morphisms by the same underlying morphism in E. Its image is the full subcategory of $(p \downarrow b)$ spanned by objects $(e, 1_b)$ with $e \in E_b$.

Recall that a functor

$$u : A \to A'$$

is *final* if for every object $a' \in A'$, the comma category

$$(a' \downarrow u)$$

is nonempty and connected. Equivalently, for every cocomplete category C and every functor $G : A' \to C$, the canonical map

$$\operatorname*{colim}_{A} Gu \to \operatorname*{colim}_{A'} G$$

is an isomorphism [1, Chapter IX].

3. Fiberwise cofinality

Definition 3.1. A functor

$$p : E \to B$$

is *left fiberwise cofinal* if for every $b \in B$, the canonical functor

$$\iota_b : E_b \to (p \downarrow b)$$

is final.

Equivalently, p is left fiberwise cofinal if for every object

$$(e, \beta : p(e) \to b)$$

of $(p \downarrow b)$, the category

$$((e, \beta) \downarrow \iota_b)$$

is nonempty and connected.

Unwinding the definition, an object of

$$((e, \beta) \downarrow \iota_b)$$

is a pair (x, α), where $x \in E_b$ and

$$\alpha : e \to x$$

is a morphism of E satisfying

$$p(\alpha) = \beta.$$

A morphism

$$(x, \alpha) \to (x', \alpha')$$

is a vertical morphism

$$v : x \to x'$$

in E_b such that

$$v\alpha = \alpha'.$$

Thus left fiberwise cofinality says that every arrow $\beta : p(e) \to b$ can be lifted from e to some object over b, and that the category of such lifts is connected. This resembles the lifting condition for an opfibration, but it is weaker: no universal or cocartesian lift is required.

4. The canonical comparison

Let $p : E \to B$ be a functor, let $F : E \to C$, and assume C has the relevant colimits.

For $b \in B$, define

$$(p_! F)(b) = \operatorname*{colim}_{e \in E_b} F(e),$$

with colimiting cocone

$$\lambda^b_e : F(e) \to (p_! F)(b).$$

There is a canonical comparison map

$$\theta_b : \operatorname*{colim}_{e \in E_b} F(e) \longrightarrow \operatorname*{colim}_{(e,\beta) \in (p \downarrow b)} F(e).$$

It is induced by the functor $\iota_b : E_b \to (p \downarrow b)$. Explicitly, the cocone over E_b obtained by restricting the universal cocone over $(p \downarrow b)$ factors uniquely through the fiber colimit.

If ι_b is final, then θ_b is an isomorphism. The central theorem proves the converse as well, when the comparison is required to be an isomorphism for all diagrams $F : E \to C$.

5. Characterization theorem

Theorem 5.1. *Let*

$$p : E \to B$$

be a functor between small categories. The following are equivalent.

(1) *p is left fiberwise cofinal; that is, for every $b \in B$, the functor*

$$\iota_b : E_b \to (p \downarrow b)$$

is final.

(2) *For every cocomplete category C and every functor $F : E \to C$, the pointwise left Kan extension $\operatorname{Lan}_p F$ exists and satisfies*

$$(\operatorname{Lan}_p F)(b) \cong \operatorname*{colim}_{e \in E_b} F(e)$$

naturally in b.

(3) *For every functor $K : B \to D$, every cocomplete category C, and every functor $F : E \to C$, there is a canonical natural isomorphism*

$$\operatorname{Lan}_{Kp} F \cong \operatorname{Lan}_K (p_! F),$$

where

$$(p_! F)(b) = \operatorname*{colim}_{e \in E_b} F(e).$$

Proof. We prove $(1) \Rightarrow (2)$. By the pointwise formula for left Kan extensions,

$$(\mathrm{Lan}_p F)(b) \cong \operatorname*{colim}_{(e,\beta)\in(p\downarrow b)} F(e).$$

Since E is small and C is cocomplete, this colimit exists. If p is left fiberwise cofinal, then

$$\iota_b : E_b \to (p \downarrow b)$$

is final. Hence colimits over $(p \downarrow b)$ are computed over E_b. Therefore

$$(\mathrm{Lan}_p F)(b) \cong \operatorname*{colim}_{e\in E_b} F(e).$$

This isomorphism is natural in b because it is induced by the pointwise Kan extension formula and the functoriality of postcomposition in the comma categories $(p \downarrow b)$.

We prove $(2) \Rightarrow (1)$. Fix $b \in B$. We must show that

$$\iota_b : E_b \to (p \downarrow b)$$

is final. Let

$$a = (e, \beta : p(e) \to b)$$

be an object of $(p \downarrow b)$. We show that

$$(a \downarrow \iota_b)$$

is nonempty and connected.

Take $C = \textbf{Set}$ and take the representable functor

$$F = \mathrm{Hom}_E(e, -) : E \to \textbf{Set}.$$

By hypothesis, the canonical comparison

$$\operatorname*{colim}_{x\in E_b} \mathrm{Hom}_E(e,x) \longrightarrow \operatorname*{colim}_{(y,\gamma)\in(p\downarrow b)} \mathrm{Hom}_E(e,y)$$

is a bijection.

The right-hand colimit may be described as the set of connected components of the category of elements of the functor

$$(p \downarrow b) \to \textbf{Set}, \qquad (y,\gamma) \longmapsto \mathrm{Hom}_E(e,y).$$

An object of this category of elements is a triple

$$(y, \gamma, h),$$

where

$$\gamma : p(y) \to b, \qquad h : e \to y.$$

A morphism

$$(y,\gamma,h) \to (y',\gamma',h')$$

is a morphism $m : y \to y'$ in E such that

$$\gamma' p(m) = \gamma, \qquad mh = h'.$$

The object

$$(e, \beta, 1_e)$$

therefore determines an element of the right-hand colimit.

Since the comparison map is surjective, this element is represented by some object on the fiber side. Thus there exist $x \in E_b$ and a morphism

$$\alpha : e \to x$$

such that the objects

$$(x, 1_b, \alpha) \quad \text{and} \quad (e, \beta, 1_e)$$

lie in the same connected component of the category of elements over $(p \downarrow b)$.

Along any morphism

$$(y, \gamma, h) \to (y', \gamma', h')$$

in this category of elements, the composite

$$\gamma p(h) : p(e) \to b$$

is preserved, because

$$\gamma' p(h') = \gamma' p(mh) = \gamma' p(m) p(h) = \gamma p(h).$$

Hence this composite is constant on connected components. For $(e, \beta, 1_e)$, the composite is β. For $(x, 1_b, \alpha)$, the composite is $p(\alpha)$. Since the two objects lie in the same connected component,

$$p(\alpha) = \beta.$$

Thus (x, α) is an object of $(a \downarrow \iota_b)$. Therefore $(a \downarrow \iota_b)$ is nonempty.

We now prove connectedness. Let (x, α) and (x', α') be two objects of $(a \downarrow \iota_b)$. Thus

$$x, x' \in E_b, \qquad p(\alpha) = p(\alpha') = \beta.$$

The corresponding objects

$$(x, 1_b, \alpha), \qquad (x', 1_b, \alpha')$$

of the fiber-side category of elements represent elements in

$$\operatorname*{colim}_{x \in E_b} \mathrm{Hom}_E(e, x).$$

Their images in the right-hand colimit both lie in the same connected component as $(e, \beta, 1_e)$, because there are morphisms in the category of elements

$$(e, \beta, 1_e) \to (x, 1_b, \alpha)$$

and

$$(e, \beta, 1_e) \to (x', 1_b, \alpha')$$

given by α and α', respectively.

Since the comparison map is injective, the two elements represented by (x, α) and (x', α') are equal in the fiber colimit. Therefore they are connected by a zigzag in the category of elements of

$$E_b \to \textbf{Set}, \qquad x \longmapsto \mathrm{Hom}_E(e, x).$$

But this category of elements has objects pairs $(x, \alpha : e \to x)$ with $x \in E_b$, and morphisms are vertical maps $v : x \to x'$ satisfying $v\alpha = \alpha'$. The full subcategory on those objects satisfying $p(\alpha) = \beta$ is precisely $(a \downarrow \iota_b)$. Moreover, every morphism in the fiber category is vertical, so if $p(\alpha) = \beta$, then for any vertical v,

$$p(v\alpha) = p(v)p(\alpha) = 1_b\beta = \beta.$$

Hence the zigzag stays inside $(a \downarrow \iota_b)$. Thus $(a \downarrow \iota_b)$ is connected. Therefore ι_b is final. Since b was arbitrary, p is left fiberwise cofinal.

We prove $(2) \Rightarrow (3)$. By the formal composition law for left Kan extensions,

$$\mathrm{Lan}_{Kp} F \cong \mathrm{Lan}_K(\mathrm{Lan}_p F)$$

whenever the displayed Kan extensions exist pointwise. By (2),

$$\mathrm{Lan}_p F \cong p_! F.$$

Therefore

$$\mathrm{Lan}_{Kp} F \cong \mathrm{Lan}_K(p_! F).$$

The canonical map may be described explicitly. The unit

$$F \to (\mathrm{Lan}_p F)p$$

followed by the isomorphism $\mathrm{Lan}_p F \cong p_! F$ gives

$$F \to (p_! F)p.$$

Applying Lan_K and using the universal property of Lan_{Kp} gives the comparison

$$\mathrm{Lan}_{Kp} F \to \mathrm{Lan}_K(p_! F).$$

Under (2), this comparison is an isomorphism.

Finally, $(3) \Rightarrow (2)$ follows by taking $D = B$ and $K = 1_B$. Then $Kp = p$, and $\mathrm{Lan}_{1_B}(p_! F) \cong p_! F$. Thus $\mathrm{Lan}_p F \cong p_! F$. The three conditions are equivalent. □

6. A left fiberwise cofinal functor that is not an opfibration

The following example shows that left fiberwise cofinality is strictly weaker than being an opfibration.

Let B be the category

$$0 \longrightarrow 1.$$

Let M be the Klein four group

$$M = \{1, a, b, ab\},$$

viewed as a one-object category. Let M act transitively but not freely on the two-element set

$$L = \{f, g\}$$

by letting a interchange f and g, while b fixes both f and g.

Define a category E as follows. It has two objects e and x. The hom-sets are

$$\mathrm{Hom}_E(e, e) = \{1_e\}, \qquad \mathrm{Hom}_E(x, x) = M,$$

$$\mathrm{Hom}_E(e, x) = L = \{f, g\}, \qquad \mathrm{Hom}_E(x, e) = \emptyset.$$

Composition

$$M \times L \to L$$

is given by the chosen action of M on L.

Define

$$p : E \to B$$

by

$$p(e) = 0, \qquad p(x) = 1,$$

sending $f, g : e \to x$ to the unique arrow $0 \to 1$, and sending every element of $M = \mathrm{Hom}_E(x, x)$ to 1_1.

Proposition 6.1. *The functor $p : E \to B$ is left fiberwise cofinal but is not an opfibration.*

Proof. First consider $b = 0$. Since there is no arrow $1 \to 0$ in B, the comma category $(p \downarrow 0)$ contains only $(e, 1_0)$. Thus

$$E_0 \to (p \downarrow 0)$$

is an isomorphism and hence final.

Now consider $b = 1$. The strict fiber E_1 is the one-object category with object x and endomorphism group M. The comma category $(p \downarrow 1)$ has two kinds of objects:

$$(x, 1_1) \quad \text{and} \quad (e, 0 \to 1).$$

We check finality of

$$\iota_1 : E_1 \to (p \downarrow 1).$$

For the object $(x, 1_1)$, the comma category

$$((x, 1_1) \downarrow \iota_1)$$

has objects endomorphisms $m : x \to x$ in M. Since M is a group, this category is nonempty and connected.

For the object $(e, 0 \to 1)$, the comma category

$$((e, 0 \to 1) \downarrow \iota_1)$$

has objects the arrows $f, g : e \to x$. A morphism from f to g is an element $m \in M$ such that

$$m \circ f = g.$$

Because the action of M on $L = \{f, g\}$ is transitive, this category is nonempty and connected. Thus ι_1 is final. Therefore p is left fiberwise cofinal.

We now show that p is not an opfibration. If p were an opfibration, then the arrow $0 \to 1$ and the object $e \in E_0$ would admit a p-cocartesian lift

$$\ell : e \to x$$

over $0 \to 1$. Since the only arrows $e \to x$ are f and g, ℓ is either f or g.

Suppose $\ell = f$. For ℓ to be p-cocartesian, every arrow $h : e \to x$ over $0 \to 1$ must factor uniquely through f by a vertical endomorphism of x. But both $1 \in M$ and $b \in M$ satisfy

$$1 \circ f = f, \qquad b \circ f = f,$$

and $b \neq 1$. Thus uniqueness fails. Hence f is not cocartesian. The same argument applies to g, since $b \circ g = g$. Therefore no arrow over $0 \to 1$ with domain e is cocartesian. Hence p is not an opfibration. □

This example demonstrates that fiberwise cofinality captures the colimit-computation property without requiring cocartesian universal arrows.

7. Grothendieck opfibrations are left fiberwise cofinal

Let

$$P : B \to \mathbf{Cat}$$

be a strict functor. We use the convention that the Grothendieck construction $\int P$ has objects (b, x), where $x \in P(b)$, and morphisms

$$(b, x) \to (b', x')$$

given by pairs

$$(\alpha, u),$$

where $\alpha : b \to b'$ in B and

$$u : P(\alpha)(x) \to x'$$

in $P(b')$. The projection

$$\pi : \int P \to B$$

is the associated Grothendieck opfibration.

Theorem 7.1. *The functor*

$$\pi : \int P \to B$$

is left fiberwise cofinal.

Proof. Fix $b \in B$. We show that

$$\iota_b : (\int P)_b \to (\pi \downarrow b)$$

is final.

An object of $(\pi \downarrow b)$ is a triple

$$(c, x, \beta : c \to b),$$

where $x \in P(c)$. An object of the fiber $(\int P)_b$ is an object $y \in P(b)$. The functor ι_b sends $y \in P(b)$ to

$$(b, y, 1_b).$$

Fix an object

$$(c, x, \beta : c \to b)$$

of $(\pi \downarrow b)$. The comma category

$$((c,x,\beta) \downarrow \iota_b)$$

has as objects pairs $(y,(\beta,u))$, where $y \in P(b)$ and

$$(\beta,u) : (c,x) \to (b,y)$$

is a morphism in $\int P$. By the convention above, this means precisely that

$$u : P(\beta)(x) \to y$$

is a morphism in $P(b)$.

Thus

$$((c,x,\beta) \downarrow \iota_b) \cong (P(\beta)x \downarrow P(b)).$$

The category

$$(P(\beta)x \downarrow P(b))$$

has an initial object,

$$(P(\beta)x, 1_{P(\beta)x}).$$

Therefore it is nonempty and connected. Thus ι_b is final. Since b was arbitrary, π is left fiberwise cofinal. ☐

Corollary 7.2. *Let $P : B \to$ **Cat** be strict, let $\pi : \int P \to B$ be its Grothendieck opfibration, let $K : B \to D$, and let $F : \int P \to C$. If C is cocomplete, then*

$$\mathrm{Lan}_{K\pi} F \cong \mathrm{Lan}_K(\pi_! F),$$

where

$$(\pi_! F)(b) = \operatorname*{colim}_{x \in P(b)} F(b,x).$$

Proof. This follows from Theorem 5.1 and the preceding theorem. ☐

8. The right-handed dual

Initial functors are dual to final functors. A functor

$$u : A \to A'$$

is initial if $u^{op} : A^{op} \to (A')^{op}$ is final. Equivalently, limits over A' may be computed over A.

For right Kan extensions, the pointwise formula uses the comma category

$$(b \downarrow p),$$

not $(p \downarrow b)$:

$$(\mathrm{Ran}_p F)(b) \cong \lim_{(b \downarrow p)} FQ.$$

Definition 8.1. A functor

$$p : E \to B$$

is *right fiberwise cofinal* if, for every $b \in B$, the canonical functor

$$\kappa_b : E_b \to (b \downarrow p)$$

is initial. Here κ_b sends $e \in E_b$ to $(e, 1_b)$, regarded as an object $b \xrightarrow{1_b} p(e)$.

Theorem 8.2. *For a functor*

$$p : E \to B$$

between small categories, the following are equivalent.

1. p *is right fiberwise cofinal.*
2. *For every complete category* C *and every functor* $F : E \to C$*,*
$$(\mathrm{Ran}_p F)(b) \cong \lim_{e \in E_b} F(e)$$
naturally in b*.*
3. *For every* $K : B \to D$*,*
$$\mathrm{Ran}_{Kp} F \cong \mathrm{Ran}_K(p_* F),$$
where
$$(p_* F)(b) = \lim_{e \in E_b} F(e).$$

Proof. This is the dual of Theorem 5.1. For $(1) \Rightarrow (2)$, the pointwise formula gives

$$(\mathrm{Ran}_p F)(b) \cong \lim_{(b \downarrow p)} FQ.$$

Since $\kappa_b : E_b \to (b \downarrow p)$ is initial, limits over $(b \downarrow p)$ are computed over E_b. Hence

$$(\mathrm{Ran}_p F)(b) \cong \lim_{e \in E_b} F(e).$$

For $(2) \Rightarrow (1)$, apply Theorem 5.1 to

$$p^{op} : E^{op} \to B^{op}.$$

The strict fiber of p^{op} over b is $(E_b)^{op}$, and the comma category $(p^{op} \downarrow b)$ is naturally isomorphic to $(b \downarrow p)^{op}$. Thus left fiberwise cofinality of p^{op} is exactly right fiberwise cofinality of p. The equivalence with (3) follows from the formal composition law

$$\mathrm{Ran}_{Kp} F \cong \mathrm{Ran}_K(\mathrm{Ran}_p F).$$

□

9. Split fibrations are right fiberwise cofinal

Let

$$R : B^{op} \to \mathbf{Cat}$$

be strict. We use the convention that $\int R$ has objects (b, x), where $x \in R(b)$, and morphisms

$$(b, x) \to (b', x')$$

given by pairs

$$(\alpha, u),$$

where $\alpha : b \to b'$ in B and

$$u : x \to R(\alpha)(x')$$

in $R(b)$. Thus morphisms in the total category go covariantly in the base but contravariantly in the fiber. The projection

$$p : \int R \to B$$

is the associated split fibration.

Theorem 9.1. *The functor*

$$p : \int R \to B$$

is right fiberwise cofinal.

Proof. Fix $b \in B$. An object of $(b \downarrow p)$ is a triple

$$(c, x, \beta : b \to c),$$

where $x \in R(c)$. An object of the strict fiber E_b is an object $y \in R(b)$. The canonical functor

$$E_b \to (b \downarrow p)$$

sends $y \in R(b)$ to

$$(b, y, 1_b).$$

Fix $(c, x, \beta : b \to c)$ in $(b \downarrow p)$. We must show that

$$(\kappa_b \downarrow (c, x, \beta))$$

is nonempty and connected.

An object of this comma category is an object $y \in R(b)$ together with a morphism

$$(b, y, 1_b) \to (c, x, \beta)$$

in $(b \downarrow p)$. Such a morphism is a morphism in $\int R$

$$(\beta, u) : (b, y) \to (c, x)$$

lying over β. By the convention above, this consists of a morphism

$$u : y \to R(\beta)(x)$$

in $R(b)$.

Therefore

$$(\kappa_b \downarrow (c, x, \beta)) \cong (R(b) \downarrow R(\beta)x).$$

This category has a terminal object,

$$(R(\beta)x, 1_{R(\beta)x}).$$

Therefore it is nonempty and connected. Thus κ_b is initial, and p is right fiberwise cofinal. $\square$

Corollary 9.2. *Let* $R : B^{op} \to \mathbf{Cat}$ *be strict, let* $p : \int R \to B$ *be its Grothendieck fibration, let* $K : B \to D$*, and let* $F : \int R \to C$*. If* C *is complete, then*

$$\operatorname{Ran}_{Kp} F \cong \operatorname{Ran}_K(p_* F),$$

where

$$(p_* F)(b) = \lim_{x \in R(b)} F(b, x).$$

Proof. This follows from Theorem 8.2 and the preceding theorem. ☐

10. Relation with existing work

The results in this note are elementary and should be viewed as an explicit 1-categorical repackaging of standard facts.

The equivalence between cofinality and invariance of colimits is classical. The pointwise Kan extension formula then immediately identifies the condition

$$E_b \to (p \downarrow b)$$

as the relevant one for strict-fiber computation. More general versions of this idea appear in fibrational treatments of colimit decompositions, including Peschke–Tholen's work on diagrams, fibrations, and decomposition of colimits [8]. In that setting, the present theorem is a small ordinary-categorical instance of broader fibrational decomposition principles.

In higher category theory, related conditions appear under the language of cofinal functors, smooth or proper functors, and base-change theorems, as in Cisinski's work [9]. The present note deliberately avoids those generalities and focuses only on the ordinary 1-categorical mechanism.

The coend perspective is also relevant. The formula

$$(\operatorname{Lan}_K F)(d) \cong \int^{b \in B} D(Kb, d) \cdot F(b)$$

combined with Fubini for coends gives another concise derivation of the Grothendieck-construction case. Loregian's treatment of coend calculus provides a convenient reference for this viewpoint [4].

Thus the contribution here is not a new theory, but a clean criterion, proof, and example showing exactly which finality condition underlies the familiar Fubini formula.

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