Sectorwise Strictification of Projective Categorical Symmetry Actions in Anomalous Quantum Field Theory

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(Dated: December 12, 2025)

We formulate and prove a sectorwise strictification criterion for anomalous finite symmetry actions on particle-sector categories. Let G be a finite group and let $\mathcal{P}$ be a finite semisimple $\mathbb{C}$-linear category of superselection sectors. A projective categorical G-action on $\mathcal{P}$ consists of autoequivalences T_g, coherence isomorphisms $\mu_{g,h} : T_gT_h \Rightarrow T_{gh}$, and a normalized scalar associator anomaly $\beta \in Z^3(G; U(1))$. We prove that $[\beta] \in H^3(G; U(1))$ is invariant under changes of coherence data and that the action strictifies precisely when $[\beta] = 0$. For a simple sector $p \in \mathcal{P}$, let $G_p \subseteq G$ be the stabilizer of its isomorphism class. Then the symmetry transport of p admits an honest associative categorical action on its stabilizer if and only if $[\beta|_{G_p}] = 0 \in H^3(G_p; U(1))$. When the obstruction vanishes, the remaining choices of equivariant structure form a torsor controlled by $H^2(G_p; U(1))$. This isolates, in a purely categorical form, the sectorwise obstruction familiar from anomalous symmetry action, symmetry fractionalization, and symmetry-enriched topological phases.

INTRODUCTION

The categorical language of modern quantum field theory treats symmetries not merely as groups of transformations of fields, but as topological defects and their composition laws. In this viewpoint, an ordinary finite symmetry G is represented by codimension-one topological defects labeled by $g \in G$, and the multiplication law of G is implemented by fusion of defects. When the symmetry is anomalous, the fusion is not strictly associative at the boundary theory: associativity is restored only after accounting for an invertible bulk theory.

This perspective is standard in several related frameworks. Dijkgraaf–Witten theory relates finite-group topological gauge theory to group cohomology [1]. The functorial approach to topological field theory originates with Atiyah [2] and was developed in higher-categorical form by Baez–Dolan and Lurie [3, 4]. Generalized global symmetries were introduced by Gaiotto, Kapustin, Seiberg, and Willett [5]. Relative quantum field theory identifies anomalous theories as boundary theories of invertible bulk theories [6]. Kapustin–Seiberg studied coupling QFTs to topological field theories and the resulting changes in line and surface operators [7]. Freed–Moore–Teleman formulate topological symmetry in QFT as a calculus of topological defects, including categorical and noninvertible symmetries [8]. Gaiotto–Kulp's orbifold-groupoid framework and the SymTFT program make the bulk/boundary organization of anomalous symmetries explicit [9]. In condensed-matter language, related obstructions appear in the theory of symmetry fractionalization and G-crossed braided tensor categories [10, 11].

The purpose of this paper is therefore not to claim that anomalies acting projectively on sector categories are new. Rather, the contribution is a focused and fully explicit algebraic theorem about *sectorwise strictification*. Let $\mathcal{P}$ be a category of particle or quasiparticle sectors, and suppose G acts projectively on $\mathcal{P}$ with associator anomaly $\beta \in Z^3(G; U(1))$. For a simple sector p, only the subgroup

$$G_p = \{g \in G \mid T_g(p) \cong p\} \tag{1}$$

can act internally on p. The main theorem proves that the obstruction to making this stabilizer action genuinely associative is exactly the restricted class

$$[\beta|_{G_p}] \in H^3(G_p; U(1)). \tag{2}$$

If this class vanishes, one can choose associative stabilizer transport; after this choice, inequivalent equivariant structures form an $H^2(G_p; U(1))$-torsor.

CONVENTIONS

All categories are $\mathbb{C}$-linear. By a finite semisimple $\mathbb{C}$-linear category we mean a semisimple abelian $\mathbb{C}$-linear category with finitely many isomorphism classes of simple objects, finite-dimensional Hom spaces, and simple objects having endomorphism algebra $\mathbb{C}$. Thus every object is a finite direct sum of simple objects.

Let $\mathrm{Aut}_{\mathbb{C}}(\mathcal{P})$ denote the monoidal category of $\mathbb{C}$-linear autoequivalences of $\mathcal{P}$, with tensor product given by composition of functors. For a finite group G, write $\mathbb{B}G$ for the one-object groupoid whose automorphism group is G. A categorical action of G on $\mathcal{P}$ is a monoidal functor

$$\mathbb{B}G \longrightarrow \mathrm{Aut}_{\mathbb{C}}(\mathcal{P}). \tag{3}$$

A projective categorical action is a weakening of this notion in which the coherence pentagon may close only up to a scalar.

Group cochains are normalized unless explicitly stated otherwise. We use the multiplicative differential convention

$$(\delta\lambda)(g, h, k) = \frac{\lambda(h, k)\lambda(g, hk)}{\lambda(gh, k)\lambda(g, h)} \tag{4}$$

for a normalized 2-cochain $\lambda : G^2 \to U(1)$.

PROJECTIVE CATEGORICAL ACTIONS

Definition 1. *Let G be a finite group and let $\mathcal{P}$ be a finite semisimple $\mathbb{C}$-linear category. A $U(1)$-projective categorical action of G on $\mathcal{P}$ consists of a $\mathbb{C}$-linear autoequivalence*

$T_g : \mathcal{P} \to \mathcal{P}$ for every $g \in G$, a natural isomorphism $\mu_{g,h} : T_g T_h \Rightarrow T_{gh}$ for every $g, h \in G$, and a normalized function $\beta : G^3 \to U(1)$ such that

$$\mu_{gh,k} \circ (\mu_{g,h} T_k) = \beta(g,h,k)\, \mu_{g,hk} \circ (T_g \mu_{h,k}) \tag{5}$$

for all $g, h, k \in G$. The natural isomorphisms $\mu_{g,h}$ are honest 2-morphisms in the functor category. The word projective refers only to the scalar failure of the pentagon condition.

The action is called strictifiable if there exists a normalized 2-cochain $\lambda : G^2 \to U(1)$ such that replacing $\mu_{g,h}$ by

$$\mu'_{g,h} = \lambda(g,h)\mu_{g,h} \tag{6}$$

makes the pentagon commute without scalar anomaly.

THE COCYCLE THEOREM

Theorem 1. *For every $U(1)$-projective categorical action of G on $\mathcal{P}$, the scalar anomaly β is a normalized 3-cocycle: $\delta\beta = 1$. Moreover, if $\mu'_{g,h} = \lambda(g,h)\mu_{g,h}$ for a normalized 2-cochain $\lambda : G^2 \to U(1)$, then the new scalar anomaly is*

$$\beta' = \beta(\delta\lambda)^{-1}. \tag{7}$$

Consequently $[\beta] \in H^3(G; U(1))$ is independent of the chosen coherence isomorphisms.

Proof. Fix $g, h, k, \ell \in G$. We compare the two standard composites from $T_g T_h T_k T_\ell$ to $T_{ghk\ell}$. Because the functors T_g compose strictly as functors after parenthesization is forgotten, and because every $\mu_{a,b}$ is an honest natural isomorphism, the only possible discrepancy in the two reassociation procedures is the scalar produced by repeated use of Eq. (5).

The first path across the Mac Lane pentagon contributes

$$\beta(g,h,k)\beta(g,hk,\ell)\beta(h,k,\ell). \tag{8}$$

The second path contributes

$$\beta(gh,k,\ell)\beta(g,h,k\ell). \tag{9}$$

Since both paths are composites of the same honest natural isomorphisms between the same functors, their total scalar discrepancy must agree. Hence

$$\beta(g,h,k)\beta(g,hk,\ell)\beta(h,k,\ell) = \beta(gh,k,\ell)\beta(g,h,k\ell), \tag{10}$$

which is exactly $(\delta\beta)(g,h,k,\ell) = 1$.

Now let $\mu'_{g,h} = \lambda(g,h)\mu_{g,h}$. Then

$$\mu'_{gh,k} \circ (\mu'_{g,h} T_k) = \lambda(gh,k)\lambda(g,h)\mu_{gh,k} \circ (\mu_{g,h} T_k), \tag{11}$$

$$\mu'_{g,hk} \circ (T_g \mu'_{h,k}) = \lambda(g,hk)\lambda(h,k)\mu_{g,hk} \circ (T_g \mu_{h,k}). \tag{12}$$

Using Eq. (5), we obtain

$$\begin{aligned} \beta'(g,h,k) &= \beta(g,h,k)\frac{\lambda(gh,k)\lambda(g,h)}{\lambda(g,hk)\lambda(h,k)} \\ &= \beta(g,h,k)(\delta\lambda)^{-1}(g,h,k). \end{aligned} \tag{13}$$

Thus β changes by a coboundary, and $[\beta]$ is independent of the representative μ. □

Corollary 1. *A $U(1)$-projective categorical action of G on $\mathcal{P}$ is strictifiable if and only if $[\beta] = 0 \in H^3(G; U(1))$.*

Proof. If the action strictifies, then after replacing μ by μ' the anomaly is $\beta' = 1$. By Theorem 1, $1 = \beta(\delta\lambda)^{-1}$, so $\beta = \delta\lambda$ and $[\beta] = 0$. Conversely, if $[\beta] = 0$, choose λ with $\beta = \delta\lambda$. Then the replacement $\mu'_{g,h} = \lambda(g,h)\mu_{g,h}$ has $\beta' = 1$, so the pentagon commutes strictly. □

STABILIZERS OF SIMPLE SECTORS

Let $p \in \mathcal{P}$ be a simple object.

Definition 2. *The isomorphism stabilizer of p is the subgroup*

$$G_p = \{ g \in G \mid T_g(p) \cong p \}. \tag{14}$$

This subgroup depends only on the isomorphism class of p. It does not require a choice of particular isomorphisms $T_g(p) \to p$.

To define an actual equivariant structure on p, one must choose isomorphisms

$$\varphi_g : T_g(p) \xrightarrow{\sim} p, \qquad g \in G_p. \tag{15}$$

Different choices of the φ_g differ by scalars because p is simple and $\mathrm{End}_{\mathcal{P}}(p) = \mathbb{C}$. These scalar changes produce the familiar lower-degree cochain ambiguity in symmetry fractionalization.

SECTORWISE STRICTIFICATION

Definition 3. *Let $p \in \mathcal{P}$ be simple. A strict stabilizer transport structure on p consists of isomorphisms $\varphi_g : T_g(p) \xrightarrow{\sim} p$ for $g \in G_p$ such that the restricted projective categorical action on the full replete subcategory generated by p is equivalent to an honest categorical G_p-action.*

Theorem 2 (Sectorwise strictification)**.** *Let $p \in \mathcal{P}$ be simple. The stabilizer transport of p can be strictified if and only if*

$$[\beta|_{G_p}] = 0 \in H^3(G_p; U(1)). \tag{16}$$

Proof. Restrict the projective categorical action from G to G_p. Since $g \in G_p$ implies $T_g(p) \cong p$, each T_g preserves the full replete subcategory $\langle p \rangle \subseteq \mathcal{P}$ generated by p. Because p is simple and $\mathrm{End}(p) = \mathbb{C}$, this subcategory is equivalent to $\mathrm{Vect}_{\mathbb{C}}$.

The restricted coherence maps satisfy

$$\begin{aligned} &\mu_{gh,k} \circ (\mu_{g,h} T_k) \\ &\qquad = \beta(g,h,k)\mu_{g,hk} \circ (T_g \mu_{h,k}) \end{aligned} \tag{17}$$

for all $g, h, k \in G_p$. Therefore the anomaly of the restricted projective categorical action is exactly $\beta|_{G_p} \in Z^3(G_p; U(1))$. By Corollary 1, this restricted action strictifies if and only if $[\beta|_{G_p}] = 0$. □

EQUIVARIANT STRUCTURES AND THE H^2 TORSOR

The vanishing of $[\beta|_{G_p}]$ is an existence condition. When it vanishes, strict stabilizer transport is not unique.

Proposition 1. *Assume $[\beta|_{G_p}] = 0$. Then the set of equivalence classes of strict stabilizer transport structures on p, when nonempty, is a torsor over $H^2(G_p; U(1))$.*

Proof. Since $[\beta|_{G_p}] = 0$, choose a normalized 2-cochain λ on G_p such that $\beta|_{G_p} = \delta\lambda$. Replacing $\mu_{g,h}$ by $\mu'_{g,h} = \lambda(g,h)\mu_{g,h}$ makes the restricted associator trivial.

If λ' is another normalized 2-cochain with $\beta|_{G_p} = \delta\lambda'$, then $\delta(\lambda'\lambda^{-1}) = 1$. Hence $\eta = \lambda'\lambda^{-1}$ is a normalized 2-cocycle on G_p. Conversely, if $\eta \in Z^2(G_p; U(1))$, then $\lambda\eta$ also satisfies $\delta(\lambda\eta) = \beta|_{G_p}$. Thus the set of strictifying 2-cochains is a torsor over $Z^2(G_p; U(1))$.

Changing the chosen isomorphisms $\varphi_g : T_g(p) \to p$ by scalars $a(g) \in U(1)$ changes the resulting 2-cocycle by the coboundary δa. Therefore equivalence classes are obtained by quotienting $Z^2(G_p; U(1))$ by $B^2(G_p; U(1))$, and form an $H^2(G_p; U(1))$-torsor. □

TWO-DIMENSIONAL FINITE-GROUP ANOMALIES

The preceding results are purely algebraic. We now explain how the cocycle β arises from a finite-group 't Hooft anomaly in two-dimensional QFT. Let G be a finite internal symmetry of a two-dimensional quantum field theory. A bosonic finite-group anomaly is represented by a normalized group cocycle $\omega \in Z^3(G; U(1))$. Equivalently, ω defines a three-dimensional Dijkgraaf–Witten invertible topological field theory [1]. If the two-dimensional theory is a boundary theory for this bulk, then the topological symmetry defects labeled by G fuse with associator determined by ω.

Concretely, define the monoidal category Vect_G^ω as follows. Its simple objects are δ_g, one for each $g \in G$, with fusion $\delta_g \otimes \delta_h = \delta_{gh}$. The associator

$$(\delta_g \otimes \delta_h) \otimes \delta_k \longrightarrow \delta_g \otimes (\delta_h \otimes \delta_k) \tag{18}$$

is multiplication by $\omega(g,h,k)$. The pentagon axiom for Vect_G^ω is exactly the cocycle equation $\delta\omega = 1$.

A category of boundary sectors $\mathcal{P}$ acted on by these symmetry defects is a module category for Vect_G^ω. Unwinding the module-category associativity constraint gives autoequivalences T_g, coherence isomorphisms $\mu_{g,h}$, and associator anomaly $\beta = \omega$.

Corollary 2. *Let a two-dimensional QFT have finite symmetry G and anomaly class $[\omega] \in H^3(G; U(1))$. Let p be a simple particle or boundary sector with stabilizer G_p. Then p admits strict associative stabilizer transport if and only if*

$$[\omega|_{G_p}] = 0 \in H^3(G_p; U(1)). \tag{19}$$

When this condition holds, equivalence classes of such structures form an $H^2(G_p; U(1))$-torsor.

Proof. The anomalous symmetry defect category is Vect_G^ω, whose associator is ω. Therefore the induced projective categorical action on $\mathcal{P}$ has $\beta = \omega$. Apply Theorem 2 and Proposition 1. □

DEFECT-LINKING COCYCLES IN HIGHER DIMENSIONS

For dimensions $d > 2$, there is no universal map $H^{d+1}(BG; U(1)) \to H^3(BG; U(1))$ for an ordinary finite group G. In particular, for a discrete finite group, the naive mapping space $\mathrm{Map}(S^{d-2}, BG)$ does not generally produce a natural degree-lowering transgression to H^3 in the way one might expect from informal physical pictures. Thus the higher-dimensional statement must be formulated more carefully.

The correct general principle is the following. A d-dimensional anomalous theory with finite symmetry G is a boundary condition for an invertible $(d+1)$-dimensional bulk theory A_α classified, in favorable finite cases, by an anomaly class $\alpha \in H^{d+1}(BG; U(1))$. A projective associator on a category of sectors is obtained only after choosing a linking geometry for the sectors under consideration. Such a geometry supplies a family of closed $(d+1)$-dimensional bordisms $W(g,h,k)$ with G-background fields, one for every triple $g,h,k \in G$. The bulk partition function then defines

$$\beta_\alpha(g,h,k) = A_\alpha(W(g,h,k)). \tag{20}$$

If the bordisms $W(g,h,k)$ satisfy the boundary compatibility expressing the pentagon, then $\beta_\alpha \in Z^3(G; U(1))$.

Definition 4. *A defect-linking reduction of a d-dimensional anomalous G-theory to a sector category $\mathcal{P}$ consists of a finite semisimple sector category $\mathcal{P}$, symmetry-defect transport autoequivalences $T_g : \mathcal{P} \to \mathcal{P}$, coherence isomorphisms $\mu_{g,h} : T_g T_h \Rightarrow T_{gh}$, and a normalized cocycle $\beta_\alpha \in Z^3(G; U(1))$ obtained by evaluating the invertible bulk anomaly theory on the closed $(d+1)$-dimensional bordisms implementing the associativity pentagon of defect transport.*

With this formulation, the algebraic theorem applies without further assumptions.

Theorem 3. *Given a defect-linking reduction with cocycle β_α, a simple sector $p \in \mathcal{P}$ admits strict stabilizer transport if and only if $[\beta_\alpha|_{G_p}] = 0 \in H^3(G_p; U(1))$. When this condition holds, strict stabilizer transport structures form an $H^2(G_p; U(1))$-torsor.*

Proof. This is Theorem 2 and Proposition 1 applied to the projective categorical action produced by the defect-linking reduction. □

HIGHER-FORM AND HIGHER-CATEGORICAL SYMMETRIES

For higher-form and higher-group symmetries, the correct object is not merely a group G, but a finite homotopy type such as $B\mathbb{G} = K(A, p+1)$ for a finite abelian p-form symmetry A. In this case, mapping spaces such as $\mathrm{Map}(S^q, B\mathbb{G})$ can contain genuine higher-homotopical information. For example,

$$\pi_0\mathrm{Map}(S^2, K(A,2)) \cong H^2(S^2; A) \cong A. \tag{21}$$

More precisely, for finite abelian A,

$$\mathrm{Map}(S^2, K(A,2)) \simeq K(A,2) \times A \tag{22}$$

as a homotopy type, since maps from S^2 to $K(A,2)$ are classified by $H^2(S^2; A)$, and the component of a map has residual $K(A,2)$-type.

A full theorem for higher symmetries requires more than replacing G by an ∞-group. The coherence data are indexed by higher simplices, and strictification generally involves a tower of obstructions rather than a single H^3-class. Accordingly, this paper does not assert a general higher-categorical strictification theorem. Instead, the present results should be read as the 1-categorical truncation relevant when the defect-linking construction has already produced an ordinary projective categorical action with scalar associator $\beta \in Z^3(G_{\text{eff}}; U(1))$ for some effective finite group G_{eff} acting on the sector category.

PHYSICAL CONSEQUENCES

Sector exclusion. Let $p, q \in \mathcal{P}$ be simple sectors. Suppose there is a symmetry-preserving local operator that identifies their stabilizer transport structures. Then their restricted anomaly classes must agree under the induced stabilizer identification. Thus, if no group isomorphism between stabilizers sends $[\beta|_{G_p}]$ to $[\beta|_{G_q}]$, then no symmetry-preserving process can identify p and q. This is a necessary condition, not a complete classification of morphisms between sectors.

Fractionalization data. If $[\beta|_{G_p}] = 0$, then sector p can carry an honest stabilizer action, but the choice is not unique. The residual ambiguity is classified by $H^2(G_p; U(1))$. This is the categorical version of the usual symmetry-fractionalization pattern: degree-three data obstruct existence, while degree-two data classify choices once the obstruction vanishes [10, 11].

Gauging obstruction. Gauging a finite symmetry requires coherent summation over symmetry backgrounds. If the projective action on the full sector category has nonzero $[\beta] \in H^3(G; U(1))$, then the symmetry cannot be gauged without either coupling to a compensating bulk theory or extending the system by additional degrees of freedom. Sectorwise, a sector p can participate in an honest stabilizer-equivariant subsector only when $[\beta|_{G_p}] = 0$.

COMPARISON WITH EXISTING FRAMEWORKS

The results above are compatible with, and should be viewed as a sectorwise refinement of, several established frameworks. Dijkgraaf–Witten theory identifies finite-group topological actions with group-cohomology cocycles [1]. In two dimensions, the anomalous symmetry defect category Vect_G^ω has associator ω, and the sectorwise theorem becomes a direct statement about module categories over Vect_G^ω.

Freed–Moore–Teleman formulate topological symmetry in QFT in terms of topological defects and their functorial calculus [8]. The present theorem does not replace that framework; it extracts a simple consequence at the level of a finite semisimple 1-category of sectors. Gaiotto–Kulp's orbifold groupoids and the SymTFT viewpoint organize anomalous symmetries via a higher-dimensional topological theory [9]. The present paper uses the same physical principle but focuses on the algebraic question: given the resulting scalar associator on sector transport, what is the exact obstruction to strictifying a single sector's stabilizer action?

Barkeshli–Bonderson–Cheng–Wang develop a broad theory of symmetry fractionalization, defects, and gauging in topologically ordered phases [10]. The H^3 obstruction and H^2 torsor structure appearing here are the finite semisimple categorical counterpart of that theory. Etingof–Nikshych–Ostrik classify fusion-category extensions using homotopy-theoretic obstruction theory [11]. The present theorem is much narrower: it concerns projective categorical actions on a given sector category and the restriction of the associator class to stabilizers of simple objects.

CONCLUSION

A projective categorical symmetry action carries a canonical degree-three obstruction class $[\beta] \in H^3(G; U(1))$. The global vanishing of this class is equivalent to strictifiability of the whole categorical action. The sectorwise refinement proved here says that a simple sector p only sees the restriction of this class to its stabilizer:

$$[\beta|_{G_p}] \in H^3(G_p; U(1)). \tag{23}$$

This restricted class is the exact obstruction to honest associative stabilizer transport of p. If it vanishes, the remaining choices are classified by an $H^2(G_p; U(1))$-torsor. The result is intentionally modest but precise: it supplies a clean algebraic bridge between anomalous categorical symmetry actions and sectorwise symmetry fractionalization.

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[1] R. Dijkgraaf and E. Witten, Topological gauge theories and group cohomology, Commun. Math. Phys. **129**, 393 (1990).

[2] M. F. Atiyah, Topological quantum field theory, Publ. Math. IHES **68**, 175 (1988).

[3] J. C. Baez and J. Dolan, Higher-dimensional algebra and topological quantum field theory, J. Math. Phys. **36**, 6073 (1995).

[4] J. Lurie, On the Classification of Topological Field Theories, Current Developments in Mathematics **2008**, 129 (2009).

[5] D. Gaiotto, A. Kapustin, N. Seiberg, and B. Willett, Generalized global symmetries, J. High Energy Phys. **2015**, 172 (2015).

[6] D. S. Freed and C. Teleman, Relative quantum field theory, Commun. Math. Phys. **326**, 459 (2014).

[7] A. Kapustin and N. Seiberg, Coupling a QFT to a TQFT and duality, J. High Energy Phys. **2014**, 001 (2014).

[8] D. S. Freed, G. W. Moore, and C. Teleman, Topological symmetry in quantum field theory, Quantum Topol. **15**, 779 (2024).

[9] D. Gaiotto and J. Kulp, Orbifold groupoids, J. High Energy Phys. **2021**, 132 (2021).

[10] M. Barkeshli, P. Bonderson, M. Cheng, and Z. Wang, Symmetry fractionalization, defects, and gauging of topological phases, Phys. Rev. B **100**, 115147 (2019).

[11] P. Etingof, D. Nikshych, and V. Ostrik, Fusion categories and homotopy theory, Quantum Topol. **1**, 209 (2010).